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AUREL WINTNER, 1903-1958.

With deep sorrow we announce the sudden and premature death of our colleague, Aurel Wintner, on January 15, 1958.

Aurel Wintner was born in Budapest on April 8, 1903. He studied mathematics and astronomy at the University of Leipzig. During that period he was editorial assistant for the *Mathematische Zeitschrift* under Leon Lichtenstein, who was then, and for many years, chief editor of that journal; this period of apprenticeship had a profound influence on him, and he always preserved a deep sense of gratitude to the man under whom he had served. He took his Ph. D. in Leipzig in 1929, and then spent one semester in Rome as a fellow of the Rockefeller Foundation and another in Copenhagen, where he worked with E. Strömgren. In 1929 he published in Leipzig the book "Spektraltheorie der unendlichen Matrizen," which established his reputation as a mathematician. It was in Copenhagen that, in the year 1930, he married the daughter of Otto Hölder, one of his Leipzig teachers. In 1930 he came to the United States to join the faculty of The Johns Hopkins University, where he remained until his death. In the year 1944 he became one of the editors of this Journal, to which he devoted the best part of his energy, both by his scientific contributions (a substantial part of his most valuable work since he came to America was published there) and as an editor. For many years he was the mainspring of the *American Journal of Mathematics*. In a very real sense, he was the Journal.

THE EDITORS

254689

ON A CONNECTEDNESS THEOREM FOR A BIRATIONAL TRANSFORMATION AT A SIMPLE POINT.*

By J. P. MURRE.[1]

Introduction. In 1951, Zariski obtained as a special case of his connectedness theorems the fact that the total transform of a normal point by a birational transformation is connected [III]. In a recent paper, [IV], the connectedness of the total transform of a simple point by a birational transformation is proved without using the theory of holomorphic functions developed in [III].

In this paper another proof is obtained of this connectedness-theorem for a birational transformation at a simple point. In fact, a somewhat stronger result is proved, namely, the following. Chow has introduced the concept of *linear connectedness*, a set is said to be linearly connected if every pair of points can be connected by a sequence of rational curves in that set. In this sense, the total transform of a simple point by a birational transformation is linearly connected.[2]

Furthermore, some applications are derived for specializations of a complete set of conjugates over a purely transcendental function field.

The terminology and notations are from [I].

In conclusion, I want to express my warmest thanks to Professor A. Weil and Professor T. Matsusaka for their advice and interest during the preparation of this paper. I am especially indebted to Professor Weil for suggestions which improve the exposition of the proof of Theorem 1.

1. The connectedness of the total transform of a point by a birational transformation of n-space. Let V be an (abstract) variety and let $\mathfrak{B}$ be a

* Received December 15, 1956; revised July 29, 1957.

[1] This paper is the first part of a thesis presented to the University of Leyden in The Netherlands in fulfillment of the requirements for the Ph. D. examination. The paper was prepared while the author was working at the University of Chicago as a participant of the I. C. A. Scientists Research Project TA-01-101-3006.

[2] I am indebted to Professor Chow for calling my attention to the fact that the original proof implicitly contained this result. He also gave as an example a cubic cone, which shows that the assumption of a simple point is essential for this linearity property.

bunch of subvarieties of V. The point set attached to $\mathfrak{B}$ will be denoted by $\{\mathfrak{B}\}$. $\mathfrak{B}$ is called *connected* when it is not possible to divide the components of $\mathfrak{B}$ into two new bunches $\mathfrak{C}$ and $\mathfrak{D}$ such that $\{\mathfrak{B}\} = \{\mathfrak{C}\} \cup \{\mathfrak{D}\}$ and $\{\mathfrak{C}\} \cap \{\mathfrak{D}\} = \phi$. An equivalent definition for connectedness is the following. Let K be a field over which all the components of $\mathfrak{B}$ are defined and suppose Q and Q' are arbitrary points of $\{\mathfrak{B}\}$, then there exists a sequence of points in $\{\mathfrak{B}\}$, $Q = P_1, P_2, \cdots, P_{2n+1} = Q'$, such that $P_{2i} \to P_{2i-1}$ and $P_{2i} \to P_{2i+1}$ over K for $i = 1, 2, \cdots, n$. The above sequence is called a *connecting sequence* in $\mathfrak{B}$ between Q and Q'. More geometrically, $\mathfrak{B}$ is connected if and only if it is possible to connect two arbitrary points in $\{\mathfrak{B}\}$ by curves in $\{\mathfrak{B}\}$. (This follows, for instance, from [II], Lemma 7.) The concept of connectedness can also be obtained by introducing the Zariski-topology on V and then employing the usual definition of connectedness in this topology. Further two points are said to be *linearly connected in* $\mathfrak{B}$ if there exists a sequence of *rational* curves in $\{\mathfrak{B}\}$ connecting them, and $\mathfrak{B}$ is said to be *linearly connected* if every pair of its points are linearly connected in $\mathfrak{B}$.

Suppose U and V are varieties; let W be a subvariety of $U \times V$ and let Q be a point of U. If there is any component in $W \cap (Q \times V)$ then this component is of the form $Q \times Z$, ([I], Chap. IV, Cor. 6 of Th. 6), where Z is a subvariety of V. The bunch $W\{Q\}$, the *total transform* of Q by W, is defined as follows: $W \cap (Q \times V) = Q \times W\{Q\}$.

To prove the connectedness theorem, we first prove a—rather formal—lemma.

LEMMA.[3] *Let $f_0, f_1, \cdots, f_m$ be $m+1$ polynomials in the indeterminates $V_1, \cdots, V_h$ and Z with coefficients in a field k. Let $v_1, \cdots, v_h$ and z be $h+1$ elements of the ring $k[[T]]$ such that the $f_i(v, 0)$ are not all 0 and the $f_i(v, z)$ are not all 0. Then there is a field $K \supset k$ and an integer $N > 0$ and elements z_j $(j = 0, 1, \cdots, 2n;\ z_0 = z,\ z_{2n} = 0)$ in the ring $K[[T^{1/N}]]$, with the following properties. For each j, not all the $f_i(v, z_j)$ are 0, and if we denote by R_j the point in projective m-space obtained by putting $T = 0$ in the ratios of the homogeneous coordinates of the point $[f_0(v, z_j), \cdots, f_m(v, z_j)]$, then for every triple $(2q, 2q+1, 2q+2)$, where $q = 0, 1, \cdots, n-1$, either $R_{2q} = R_{2q+1} = R_{2q+2}$ or there exists a rational curve C_q having, over a suitable field containing k, the point R_{2q+1} as a generic point and going through R_{2q} and R_{2q+2}. Furthermore, if z is 0 for $T = 0$ then the same holds for all z_j.*

Proof. Let $f_i(V, Z) = \sum_{j=0}^{M} f_{ij}(V) Z^j$. The lemma will be proved by induc-

[3] In the following $k[[T]]$ is the power-series ring in a letter T over the field k.

tion on the integer $\min._i \operatorname{order}_T f_i(v,z) - \min._{ij} \operatorname{order}_T[f_{ij}(v)z^j]$; the proof will be divided into two parts.

A. Let $\min._i \operatorname{order}_T f_i(v,z) - \min._{ij} \operatorname{order}_T[f_{ij}(v)z^j] = 0$. In order to simplify notation, write $\lambda_j = \min._i \operatorname{order}_T f_{ij}(v)$ and $\mu_j = \min._i \operatorname{order}_T[f_{ij}(v)z^j]$. If $r = \operatorname{order}_T z$, then $\mu_j = \lambda_j + rj$. With the above assumption, we have $\min._j \mu_j = \min._i \operatorname{order}_T f_i(v,z)$. There is an index q $(0 \leqq q \leqq M)$ uniquely determined by the conditions

$$\mu_q = \min._j \mu_j \text{ and } \mu_j > \mu_q \text{ for } j > q. \tag{1}$$

If $q = 0$, the sequence of elements mentioned in the conclusions of the lemma is obtained by taking $z_0 = z$, $z_1 = z$, $z_2 = 0$; then for the points R_j we have $R_0 = R_1 = R_2$. Next we will proceed by induction on the integer q.

Clearly, the proof of part **A.** will be completed if it is possible to construct elements z^* and z' of $k'[[T']]$, where T' is a fractional power of T and where $k' \supset k$ and:

a. Not all the $f_i(v,z^*)$ are 0 and not all the $f_i(v,z')$ are 0. If R, R^* and R' are the points obtained (as explained in the conclusions of the lemma) by substituting $T' = 0$ in the ratios of the $f_i(v,z)$, the $f_i(v,z^*)$ and the $f_i(v,z')$, then the locus of R^* over k is a rational curve and R and R' are points on this curve.

b. If $\mu'_j = \min._i \operatorname{order}_{T'}[f_{ij}(v)z'^j]$, then $\min._j \mu'_j = \min._i \operatorname{order}_{T'} f_i(v,z')$.

c. If q' is such that $\mu'_{q'} = \min._j \mu'_j$ and $\mu'_j > \mu'_{q'}$ for $j > q'$, then $q' < q$.

d. If $\operatorname{order}_T z$ is positive, then the orders in T' of z^* and z' are positive.

In order to construct elements z^* and z', consider the expressions[4] $(\lambda_j - \lambda_q)/(q-j)$ for $j < q$. There is a uniquely determined integer q' $(0 \leqq q' < q)$ such that

$$\begin{cases} (\lambda_{q'} - \lambda_q)/(q-q') \text{ is the minimum of the expressions} \\ (\lambda_j - \lambda_q)/(q-j) \text{ for } j < q \text{ and } (\lambda_j - \lambda_q)/(q-j) > \\ (\lambda_{q'} - \lambda_q)/(q-q') \text{ for } j < q'. \end{cases} \tag{2}$$

[4] This expression arises in the following way: What should the order in T of z be, if we want, for a certain fixed $j<q$, $\min._i \operatorname{order}_T[f_{ij}(v)z^j] = \min._i \operatorname{order}_T[f_{iq}(v)z^q]$? The answer is $(\lambda_j - \lambda_q)/(q-j)$. Since $\mu_j = \lambda_j + rj$ and $\mu_q = \min._j \mu_j$, we have that $r \leqq \min._{j<q}(\lambda_j - \lambda_q)/(q-j)$. If now z^* is an element of $k[[T]]$ whose order (in T) is precisely this minimum, then there remains, for some i, a term $f_{iq}(v)z^{*q}$ among the terms of lowest order in T.

Let $T' = T^{1/(q-q')}$; if $\lambda'_j = \min._i \operatorname{order}_{T'} f_{ij}(v)$ then $\lambda'_j = \lambda_j(q-q')$. Suppose $z = \alpha_r T^r +$ higher powers in T ($\alpha_r \in k$); take two independent transcendentals over k, β^* and β', write $k' = k(\beta^*, \beta')$ and let $z^* = \beta^* T'^{(\lambda_{q'}-\lambda_q)/(q-q')}$, i.e., $z^* = \beta^* T'^{(\lambda_{q'}-\lambda_q)}$. If

$$\mu^*_j = \min._i \operatorname{order}_{T'}[f_{ij}(v) z^{*j}] \text{ then } \mu^*_j = \lambda_j(q-q') + j(\lambda_{q'} - \lambda_q).$$

By the assumptions, not all the λ_j are infinite, hence it follows that not all the μ^*_j are infinite. Since β^* is transcendental over k, it follows that not all the $f_i(v, z^*)$ are zero. Furthermore,

$$\begin{aligned}\mu^*_j - \mu^*_q &= (\lambda_j - \lambda_q)(q-q') + (j-q)(\lambda_{q'} - \lambda_q) \\ &= (\mu_j - \mu_q)(q-q') + (j-q)(\mu_{q'} - \mu_q).\end{aligned}$$

It follows from this by (1) and (2) that

(3) $\qquad \mu^*_{q'} = \mu^*_q = \min._j \mu^*_j;\ \mu^*_j > \mu^*_{q'}$ for $j > q$ and for $j < q'$.

Now let R^* and R be the above mentioned points (see **a.**). Since β^* is transcendental over k, it follows that the homogeneous coordinates of R^* are polynomials in β^* with coefficients in k. From the relations (1), (2) and (3) it follows that $R^* \to R$ over k (take $\beta^* \to \alpha_r$ if $\mu_q = \mu_{q'}$, and $\beta^* \to \infty$ if $\mu_q < \mu_{q'}$).

Next we will make the assumption that the index q' fulfills the condition

(4) $\qquad q' < \min._{j<q'}(\mu^*_j - \mu^*_{q'}).$

This assumption can be made; for by the inequalities of (3), this minimum is positive and it can then be made arbitrarily large by replacing T' by a power of T'. Now let $z' = \beta' T'^{(\lambda_{q'}-\lambda_q)+1}$; if $\mu'_j = \min._i \operatorname{order}_{T'}[f_{ij}(v) z'^j]$ then $\mu'_j = \mu^*_j + j$. From this, in combination with (3) and (4), it follows that

(5) $\qquad \mu'_{q'} = \min._j \mu'_j;\ \mu'_j > \mu'_{q'}$ for $j > q'$ and $j < q'$.

Since by construction $q' < q$, it follows that z' fulfills condition **c**. Again, since β' is transcendental over k, it follows that not all the $f_i(v, z')$ are zero; therefore, it is possible to construct a point R' by putting $T' = 0$ in the ratios of the $f_i(v, z')$. Further, since β' is transcendental over k, it also follows that $\mu'_{q'} = \min._j \mu'_j = \min._i \operatorname{order}_{T'} f_i(v, z')$; therefore, z' fulfills condition **b**. Next, we want to show that $R^* \to R'$ over k. It follows from the inequalities in (5) that R' is the point obtained by substituting $T' = 0$ in the ratios of the homogeneous coordinates of the point $[f_{0q'}(v), \cdots, f_{mq'}(v)]$. As we have seen above, the homogeneous coordinates of the point R^* are polynomials in β^* with coefficients in k; by (3) these polynomials begin

with terms of degree q' in the β^* (and end with terms of degree q). Substituting $\beta^* = 0$ in the homogeneous coordinates of R^* therefore leads to the same point as substituting $T' = 0$ in the homogeneous coordinates of the point $[f_{0q'}(v), \cdots, f_{mq'}(v)]$, i.e., to the point R' as we have just seen. Therefore, $R^* \to R'$ over the specialization $\beta' \to 0$ with respect to k, and hence **a.** is satisfied by z^* and z'. It remains to be shown that **d.** is also fulfilled. As before, write $z = \alpha_r T^r + \cdots$. From the fact that $\mu_q \leqq \mu_{q'}$ (by (1)) and that $\mu_j = \lambda_j + jr$, it follows that $r \leqq (\lambda_{q'} - \lambda_q)/(q - q')$. Since $T = T'^{(q-q')}$, $z^* = \beta T'^{(\lambda_{q'}-\lambda_q)}$ and $z' = \beta' T'^{(\lambda_{q'}-\lambda_q)+1}$, it follows that

$$\operatorname{order}_{T'} z \leqq \operatorname{order}_{T'} z^* \leqq \operatorname{order}_{T'} z'.$$

Hence **d.** is satisfied, which completes the proof of **A.**

B. Assume that the lemma is proved if the difference $\min._i \operatorname{order}_T f_i(v, z) - \min._{ij} \operatorname{order}_T [f_{ij}(v) z^j]$ is smaller then some integer d. Now suppose this difference is equal to d.

Let R and S be the points obtained by substituting $T = 0$ in the ratios of the $f_i(v, z)$ and the $f_i(v, 0)$, respectively. We adopt the following terminology during the rest of the proof of this lemma to simplify notation. If z and z' are elements of a ring $K[[T]]$ such that not all the $f_i(v, z)$ are 0 and not all the $f_i(v, z')$ are 0, and if R, respectively R', are the points obtained by substituting $T = 0$ in the ratios of the $f_i(v, z)$, respectively the $f_i(v, z')$, then we will call a sequence of points $R = R_0, R_1, \cdots, R_j, \cdots, R_{2n} = R'$ a *connecting sequence of points* between R and R' if the following are true: Every point R_j is obtained by means of an element z_j of $K[[T]]$ and a substitution $T = 0$ in the ratios of the $f_i(v, z_j)$ (which implies that not all the $f_i(v, z_j)$ are 0), and both R_{2q} and R_{2q+2} are specializations of R_{2q+1} over k for $q = 0, 1, \cdots, n-1$ (it is a tacit agreement that K contains k; it should be noted that in the present terminology—which is only used during the proof of this lemma—the field k plays a particular role). Further, if z and z' are 0 for $T = 0$ then we require the same for all the z_j. Moreover, we will call such a sequence $R = R_0, R_1, \cdots, R_{2n} = R'$ a *linearly connecting sequence of points* between R and R' if, for every triple $(2q, 2q+1, 2q+2)$ of this sequence, either $R_{2q} = R_{2q+1} = R_{2q+2}$ or there is a rational curve C_q having R_{2q+1} as a generic point over a suitable field containing k, and going through R_{2q} and R_{2q+2}.

Write $Z = X + Y$, let

$$f_i(V, Z) = \sum_{j=0}^{M} f_{ij}(V) Z^j = \sum_{j=0}^{M} g_{ij}(V, X) Y^j = g_i(V, X, Y).$$

(Note that $f_i(V,X) = g_{i0}(V,X)$.) Let $z = \alpha_r T^r + \alpha_{r+1}T^{r+1} + \alpha_{r+2}T^{r+2} + \cdots$, where $\alpha_r, \alpha_{r+1}, \cdots \in k$. Let β be transcendental over k and take $x = \alpha_r T^r + \beta T^{r+1}$. Now, by the assumptions not all the $f_i(v,0)$ are zero, therefore, since β is transcendental over k, not all the $f_i(v,x) = g_{i0}(v,x)$ are zero. Let R_I be the point obtained by substituting $T=0$ in the ratios of the $f_i(v,x)$. Next let γ be transcendental over $k(\beta)$ and let $y = T^r(-\alpha_r + \gamma - \beta T)$, i.e., $x + y = \gamma T^r$. Since γ is transcendental over k, it follows that

$$\min._i \operatorname{order}_T f_i(v, x+y) = \min._{ij} \operatorname{order}_T [f_{ij}(v)(x+y)^j]. \tag{6}$$

The right hand side of (6) being finite implies that not all the $f_i(v,x+y)$ are 0, let R_{II} be the point obtained by substituting $T=0$ in the ratios of the $f_i(v,x+y)$. It follows by (6) and the special case of the lemma proved under **A.** that (in the just mentioned terminology) there is a linearly connecting sequence of points between R_{II} and S. We will show next that there is also a linearly connecting sequence of points between R_{II} and R_I. Consider the polynomials $g_i(V,X,Y)$ in the indeterminates $V_1, \cdots, V_h$, X and Y and consider the elements $v_1, \cdots, v_h$, x and y of $k(\beta,\gamma)[[T]]$. Since γ is transcendental over $k(\beta)$ it follows that

$$\min._i \operatorname{order}_T g_i(v,x,y) = \min._{ij} \operatorname{order}_T [g_{ij}(v,x)y^j].$$

R_{II} and R_I being the points obtained by substituting $T=0$ in the ratios of the $f_i(v,x+y) = g_i(v,x,y)$ and the $f_i(v,x) = g_i(v,x,0)$ respectively, it follows therefore by the special case of the lemma proved under **A.** that there is a linearly connecting sequence of points between R_{II} and R_I. (Note that if $\operatorname{order}_T z$ is positive then the orders in T of x, y and $x+y$ are positive, and therefore, by the special case of the lemma proved under **A.**, every intermediate point R_j is obtained by means of a z_j which has positive order in T.)

From the facts proved above, it follows that it is sufficient to prove that there is a linearly connecting sequence of points between R and R_I; the points R, respectively R_I, are obtained by putting $T=0$ in the $f_i(v,z)$, respectively the $f_i(v,x) = g_i(v,x,0)$. Now, let

$$y' = (\alpha_{r+1} - \beta)T^{r+1} + \alpha_{r+2}T^{r+2} + \cdots, \text{ i.e., } z = x + y';$$

then R can also be obtained by substituting $T=0$ in the $f_i(v,x+y') = g_i(v,x,y')$. If μ_q is the integer introduced in (1), i.e., $\min._{i,j} \operatorname{order}_T [f_{ij}(v)z^j]$, then it follows from the way in which the x and y' are chosen that the terms of degree μ_q in T have cancelled in the $g_i(v,x,y')$, and hence

$$\min._{ij} \operatorname{order}_T [f_{ij}(v)z^j] < \min._{ij} \operatorname{order}_T [g_{ij}(v,x)y'^j].$$

Since $f_i(v,z) = g_i(v,x,y')$, we then have that

$$\min._i \operatorname{order}_T g_i(v,x,y') - \min._{ij} \operatorname{order}_T [g_{ij}(v,x)y'^j]$$

is smaller than $\min._i \operatorname{order}_T f_i(v,z) - \min._{ij} \operatorname{order}_T [f_{ij}(v)z^j]$. We can now apply the induction hypothesis to the polynomials $g_i(V,X,Y)$ (instead of the $f_i(V,Z)$), to the elements $v_1, \cdots, v_h, x$ (instead of the $v_1, \cdots, v_h$) and y' (instead of the z) and to the points R and R_I obtained, respectively, by putting $T=0$ in the ratios of the $g_j(v,x,y')$ and the $g_i(v,x,0)$. This completes the proof of the lemma.

THEOREM 1. *Let W be a subvariety of $S^n \times P^m$ (n-space and projective m-space). Suppose the projection of W on S^n is equal to S^n itself, and let $[W: S^n] = 1$. Let Q be a point of S^n. Then $W\{Q\}$ is linearly connected.*

Proof. We will proceed by induction on the integer n; for $n=1$ the theorem is clearly true.

Let k be a field of definition for W. Without loss of generality, it may be assumed that Q is the origin O of S^n and that k is algebraically closed, so that, in particular, the components of $W\{O\}$ are defined over k.

Let (x,y) be a generic point of W over k, where $(x) = (x_1, \cdots x_n)$ and where $(y) = (y_0, \cdots, y_m)$ are homogeneous coordinates in P^m, with $y_i = f_i(x)$, $f_i(X) \in k[X_1, \cdots, X_n]$. Write

(1) $f_i(X) = \sum_{j=s}^{t} f_{ij}(X)$ with $f_{ij}(X)$ homogeneous of degree j and for at least one i, $f_{is}(X)$ is different from zero.

First we will fix, in a certain way, a point R_0 in $W\{O\}$, and then we show that every point in $W\{O\}$ is linearly connected in $W\{O\}$ with this point R_0. We fix this point R_0 as follows. Let the quantities u_i be defined by the equations

(2) $$x_i = u_i x_1, \qquad i = 2, 3, \cdots, n.$$

Let $(x,u) \to (0,u')$ over k, with u'_i independent transcendentals over k. There exists an index j_0 such that $f_{j_0 s}(X) \neq 0$, the quotients y_j/y_{j_0} then have a unique specialization over the specialization $(x,u) \to (0,u')$ with reference to k; this specialization determines a point R_0 in $W\{O\}$. We note that R_0 depends only on the $f_{is}(X)$ and on the specialization of the ratios of the x_i, and not on the particular index 1 chosen to define the quantities u_i.

Now suppose there is given a point R in $W\{O\}$, we will show that R is linearly connected with R_0 in $W\{O\}$. Let $k(R)$ be the field obtained

from adjoining to k the non-homogeneous coordinates of R. It is possible to construct a field K, containing $k(R)$, and a homomorphism $\phi: k[x] \to K[[T]]$, leaving k invariant, such that the point R can be obtained from substituting $T=0$ in the ratios of the $\phi[f_i(x)]$ (this implies, in particular, that not all the $\phi[f_i(x)]$ are 0); moreover, this ϕ can be constructed in such a way that not all the $\phi[f_{is}(x)]$ are 0. (The construction of such a field K and homomorphism ϕ can be done in the following way. Take a linear space L^{m+1} generic through the point $O \times R$, then $W \cap L$ consists of curves C_i [see, for instance, the proof of Prop. 7 of App. II of [I]]; let C_1 go through the point $O \times R$. Let K be the algebraic closure of the field which is obtained from adjoining to $k(R)$ the coefficients of a set of equations of L. A generic point (x) of C_1 over K is a generic point of W over k [see again [I], App. II, proof of Prop. 7]. Take a valuation on C_1 with center $O \times R$, this determines a homomorphism of $K[x] \to K[[T]]$; the restriction to $k[x]$ fulfills all the requirements.)

Given the point R, we construct such a field K and homomorphism $\phi: k[x] \to K[[T]]$. Among the $x_1, \cdots, x_n$, for instance x_1 starts with a term of lowest degree in T. Introducing the u_i as in (2), ϕ can now be extended to $k[u, x_1] \to K[[T]]$. Let the extension still be denoted by ϕ. To distinguish x_1 from the other x_i write $x_1 = z$, then we have

$$x_1 = z, \qquad x_i = u_i z; \qquad i = 2, 3, \cdots, n. \tag{3}$$

Note that z is transcendental over $k(u)$; therefore, the specialization of z over every specialization of the $u_2, u_3, \cdots, u_n$ with respect to k may be changed arbitrarily.

Now apply the lemma to the polynomials $\sum_{j=s}^{t} f_{ij}(U) Z^{j-s}$ (s was introduced in (1)) in the indeterminates $U_2, \cdots, U_n$ and Z with coefficients in the field K (which is analogous to the field k in the lemma), and to the elements $\phi(u_2), \cdots, \phi(u_n)$ and $\phi(z)$ in the ring $K[[T]]$. The homomorphism ϕ was constructed in such a way that not all the $\phi[f_{is}(u)] = f_{is}[\phi(u)]$ are 0, R was the point obtained by putting $T=0$ in the ratios of the $f_i[\phi(x)]$, i.e., in the ratios of the $\sum_{j=s}^{t} f_{ij}[\phi(u)]\phi(z)^{j-s}$; let S be the point obtained by putting $T=0$ in the ratios of the $\phi[f_{is}(u)]$. By the lemma, there is a sequence of points $R = R_0, R_1, \cdots, R_{2p} = S$, such that, for every triple $(2q, 2q+1, 2q+2)$, either $R_{2q} = R_{2q+1} = R_{2q+2}$ or there is a rational curve C_q, going through R_{2q} and R_{2q+2}, and having R_{2q+1} as a generic point over some field containing K (and hence, a fortiori containing k, over which the

components of $W\{O\}$ are defined). Moreover, since $\phi(z) = 0$ for $T = 0$, it follows by the last conclusion of the lemma and by the relations (3) that all the points R_j are in $W\{O\}$. Therefore, the sequence of points between R and S is a linearly connecting sequence of points in $W\{O\}$.

R_0 being constructed as in the beginning of the proof by means of a generic specialization of the ratios of the x_i, both R_0 and S are specializations over k of the point T with homogeneous coordinates $[f_{0s}(x), \cdots, f_{ms}(x)]$ in P^m, where $f_{is}(X)$ is homogeneous of degree s (see (1)). Let X be the locus of R_0 over k. Since R_0 is constructed by means of a generic specialization of the ratios of the x_i, it follows that X is also the locus of the point T over k; in particular $S \in X$. Since the components of $W\{O\}$ are defined over k (which is algebraically closed), it follows that X is in a component of $W\{O\}$. In order to complete the proof, it suffices to show that S and R_0 are linearly connected in X. Consider the locus Y over k of the point with coordinates $[x_1, \cdots, x_n;\ f_{0s}(x), \cdots, f_{ms}(x)]$ where the $(x_1, \cdots, x_n)$ are considered as homogeneous coordinates in P^{n-1} and, as before, $[f_{0s}(x), \cdots, f_{ms}(x)]$ are homogeneous coordinates in P^m. Then the variety Y in $P^{n-1} \times P^m$ is a birational transformation of P^{n-1}; hence, by our induction hypothesis (see beginning of the proof), the total transform by Y of every point of P^{n-1} is linearly connected. By this, the projection of Y on P^m is linearly connected; however, this projection is X. This completes the proof.

2. Multiplicities of maximal connected bunches in specializations of complete sets of conjugates. Let U and V be varieties and let W be a subvariety of $U \times V$. Suppose the projection of W on U is equal to U itself and let $\dim. W = \dim. U$. Let k be a field of definition for U, V and W and let Q be a generic point of U over k. It follows easily from [I], Chap. IV, Th. 6, that every point of $W \cap (Q \times V)$ is generic on W over k and the points of this intersection are conjugates of each other over $k(Q)$. Further, all the different conjugates over $k(Q)$ occur among the points of this intersection.

THEOREM 2. *Let W be a subvariety of $S^n \times P^m$ (n-space and projective m-space), let $\dim. W = n$ and suppose the projection of W on S^n is equal to S^n itself. Let k be a field of definition for W and let $Q \times R$ be a generic point of W over k. Let $(R^{(1)}, \cdots, R^{(g)})$ be a complete set of conjugates of R over $k(Q)$. Suppose Q' is a point of S^n and let $\mathfrak{B}$ be a maximal connected bunch of $W\{Q'\}$. Then, in every specialization of $(R^{(1)}, \cdots, R^{(g)})$ over the specialization $Q \to Q'$ with reference to k, the*

multiplicity of the bunch $\mathfrak{B}$ is the same. (The multiplicity of $\mathfrak{B}$ in a specialization $(R'^{(1)}, \cdots, R'^{(g)})$ of $(R^{(1)}, \cdots, R^{(g)})$ over the specialization $Q \to Q'$ with reference to k is defined as the number of points $R'^{(\eta)}$ which are in $\{\mathfrak{B}\}$.) Further, if $\mathfrak{B}$ occurs with multiplicity 1, then $\mathfrak{B}$ is linearly connected.

Proof. On W, consider the cycle $\sum_{\eta=1}^{g} Q \times R^{(\eta)}$ and let T be its Chow-point. The ratios of the coordinates of T are in $k(Q)$. Let Z be the locus of (Q, T) over k; Z fulfills the conditions of Theorem 1. Hence $Z\{Q'\}$ is linearly connected. Let $(Q', R'^{(1)}, \cdots, R'^{(g)})$ and $(Q', R''^{(1)}, \cdots, R''^{(g)})$ be two specializations of $(Q, R^{(1)}, \cdots, R^{(g)})$ over k and let T' and T'' be the Chow-points of $\sum_{\eta=1}^{g} Q' \times R'^{(\eta)}$ and $\sum_{\eta=1}^{g} Q' \times R''^{(\eta)}$ respectively. Let K be the algebraic closure of the field $k(Q')$; the components of $Z\{Q'\}$ and $W\{Q'\}$ are defined over K. T' and T'' are points in $Z\{Q'\}$; since $Z\{Q'\}$ is connected, it follows that there is a sequence of points T'_j $(j = 0, 1, \cdots, 2p;\ T'_0 = T',\ T'_{2p} = T'')$ in the components of $Z\{Q'\}$ such that T'_{2i} and T'_{2i+2} are both specializations of T'_{2i+1} over K for $i = 0, 1, \cdots, p-1$. Let $(R'_j{}^{(1)}, \cdots, R'_j{}^{(g)})$ be a specialization of $(R^{(1)}, \cdots, R^{(g)})$ over the specialization $(Q, T) \to (Q', T'_j)$, with reference to k, for $j = 0, 1, \cdots, 2p$. Since, by the properties of Chow-points, the cycle $\sum_{\eta=1}^{g} Q' \times R'_j{}^{(\eta)}$ is uniquely determined by the point T'_j, it follows that, for a suitable order of the points $R'_{2i}{}^{(\eta)}$, the set $(T'_{2i}, R'_{2i}{}^{(1)}, \cdots, R'_{2i}{}^{(g)})$ is a specialization of $(T'_{2i+1}, R'_{2i+1}{}^{(1)}, \cdots, R'_{2i+1}{}^{(g)})$ over K, and similarly for $2i+2$ (where $i = 0, 1, \cdots, p-1$). The components of $W\{Q'\}$ being defined over K, this completes the first part of the proof.

Assume now that $\mathfrak{B}$ has multiplicity 1 and let, for every j, $R'_j{}^{(1)}$ be the unique point in $\{\mathfrak{B}\}$. Under the present assumptions we have then $K(R'_j{}^{(1)}) \subset K(T'_j)$. The linear connectedness of $\mathfrak{B}$ then follows from the linear connectedness of $Z\{Q'\}$.

Note. Theorem 2 is a generalization of [I], Chap. III, Th. 4, except that, if $\{\mathfrak{B}\}$ reduces to a point R' (i.e., R' is a proper specialization of R over the specialization $Q \to Q'$ with reference to k), then the multiplicity of R' is a multiple of $[k(Q', R') : k(Q')]_i$. However this result can also be obtained from Theorem 2 and the theory of algebraic functions of one variable. We sketch a proof.[5] By an extension of the ground field and a change of coordinates it can be assumed that $Q' = 0$. Write $Q = (x_1, \cdots, x_n)$,

[5] Cf. [I], Chapter III, Theorem 2 and G. Shimura, *American Journal of Mathematics*, vol. 77 (1955), page 141, Prop. 13.

$R=(1, y_1, \cdots, y_m)$ $x_1=\lambda$ and $u_i=x_i/\lambda$ for $i=2,3,\cdots,n$. Consider the specialization $(x,u)\to(0,u)$ over k and let $K=k(u)$. Next consider (y) as functions of one variable λ over K. Let Ω be the completion of $K(\lambda)$ with respect to the valuation induced in $K(\lambda)$ by $\lambda\to 0$. It follows by Theorem 2 that there is a conjugate $R^{(\eta)}$ of R over $k(x)$, such that R' is a specialization of $R^{(\eta)}$ over the valuation induced by $\lambda\to 0$ in Ω. The required result then follows essentially from the fact that every specialization of $R^{(\eta)}$ over the valuation induced by $\lambda\to 0$ in Ω, is a conjugate of R' over K (since Ω is complete) and from the fact that $[K(R'):K]$ is a factor of $[\Omega(R^{(\eta)}):\Omega]$.

The following proposition is a generalization of [I], Chap. III, Prop. 9 and can be proved by the same methods.

PROPOSITION 1. *Let W be a subvariety of $S^n\times P^m$, snch that* dim. $W=n$, *and let the projection of W on S^n be equal to S^n itself. Let k be a field of definition for W. Let τ be an isomorphism of k and a field k'. Suppose Q and Q' are points of S^n such that τ can be extended to an isomorphism of $k(Q)$ and $k'(Q')$; let π be an extension of τ to the algebraic closures of $k(Q)$ and $k'(Q')$. Let $\mathfrak{B}$ be a maximal connected bunch of $W\{Q\}$. Then $\mathfrak{B}^\pi$ is a maximal connected bunch of $W^\tau\{Q'\}$. Let (x,y) and (x',y') be generic points of W and W^τ over k and k' respectively, then the multiplicities of $\mathfrak{B}$ and $\mathfrak{B}^\pi$ in the specializations of a complete set of conjugates of (y) over $k(x)$, respectively (y') over $k'(x')$, over the specialization $(x)\to Q$ with respect to k, respectively $(x')\to Q'$ with respect to k', are equal.*

THEOREM 3.[6] *Let W be a subvariety of $U\times V$, where V is a projective variety. Let* dim. $W=$ dim. U *and let the projection of W on U be equal to U. Let k be a field of definition for U, V and W and let (Q,R) be a generic point of W over k (hence Q is generic on U over k). Suppose Q' is a simple point on U and let $\mathfrak{B}$ be a maximal connected bunch of $W\{Q'\}$. Let $(R^{(1)},\cdots,R^{(\varrho)})$ be a complete set of conjugates of R over $k(Q)$. Then, for every specialization $(R'^{(1)},\cdots,R'^{(\varrho)})$ of $(R^{(1)},\cdots,R^{(\varrho)})$ over the specialization $Q\to Q'$ with respect to k, the number of points $R'^{(\eta)}$ which are in $\{\mathfrak{B}\}$ is the same. Further, if $\mathfrak{B}$ is represented only once in such a specialization, then $\mathfrak{B}$ is linearly connected.*

Proof. Suppose U^n is a variety in S^N (eventually P^N) and V^m is a variety in P^M. Let $F_i(X)$, $i=1,2,\cdots,n$, be a uniformizing set of linear forms for U at Q' with coefficients in k; by [I], Chap. IV, Prop. 25, this is a fortiori a uniformizing set of linear forms for U at Q. Write $Q=(x)$ and

[6] Theorem 3 is the analogue of [I], Chap. VI, Th. 12.

$Q' = (x')$; let L and L' be the linear varieties in S^N defined, respectively, by the equations $F_i(X - x) = 0$ and $F_i(X - x') = 0$, $i = 1, \cdots, n$. These equations can also be written in the form $F_i(X) - v_i = 0$ and $F_i(X) - v'_i = 0$ with $v_i = F_i(x)$ and $v'_i = F_i(x')$ respectively. By [I], Chap. V, Prop. 20, Q and Q' are proper components of $U \cap L$, respectively of $U \cap L'$, and of multiplicity 1. By [I], Chap. IV, Prop. 24 $k(Q)$ is algebraic (and separable) over $k(v)$. From the fact that Q' is a proper component of $U \cap L'$ of multplicity 1, it follows by [I], Chap. V, Cor. 1 of Th. 3 that Q' is a proper specialization of Q, over the specialization $(v) \to (v')$ with reference to k, of multiplicity 1. Let $(Q^{(1)}, \cdots, Q^{(d)})$ be a complete set of conjugates of Q over $k(v)$ and let $Q^{(1)} = Q$. A complete set of conjugates of (Q, R) over $k(v)$ consists of the dg points $(Q^{(\tau)}, R^{(\tau\eta)})$, $\tau = 1, \cdots, g$, where the set of points $(R^{(\tau 1)}, \cdots, R^{(\tau g)})$ is obtained from $(R^{(1)}, \cdots, R^{(g)})$ by applying to the algebraic closure of $k(v)$ an automorphism over $k(v)$ such that Q is transformed in $Q^{(\tau)}$. Consider the locus X over k of the point (v, Q, R). X is a subvariety of $S^n \times P^N \times P^M$ (after, perhaps, a projective closure of S^N); since a product of projective spaces can be embedded in an everywhere biregular way into a projective space, it follows that X fulfills the conditions of Theorem 2.

Now let a specialization $(R'^{(1)}, \cdots, R'^{(g)})$ of $(R^{(1)}, \cdots, R^{(g)})$ over the specialization $Q \to Q'$ with reference to k be given; over this specialization (v) specializes necessarily to (v'). Next, extend this specialization to a specialization of all the $(Q^{(\tau)}, R^{(\tau\eta)})$ over the specialization $(v) \to (v')$ with reference to k. If $\mathfrak{C}$ is a maximal connected bunch of $X\{v'\}$; then by Theorem 2 the number of points $(Q'^{(\tau)}, R'^{(\tau\eta)})$ which are in $\{\mathfrak{C}\}$ is always the same, independently of the way in which the specialization $(v) \to (v')$ with reference to k is extended. Clearly, among the bunches $\mathfrak{C}$ of $X\{v'\}$ there must be one which contains (as point set) $Q' \times \{\mathfrak{B}\}$; however, since Q' is a proper specialization of Q, over $(v) \to (v')$ with reference to k, it follows that $Q' \times \{\mathfrak{B}\}$ itself is such a bunch $\{\mathfrak{C}\}$ (i.e., a maximal connected bunch of $X\{v'\}$). Now let $(Q'^{(\tau)}, R'^{(\tau\eta)})$ be in $Q' \times \{\mathfrak{B}\}$; since Q' is a proper specialization of Q, over $(v) \to (v')$ with reference to k, of multiplicity 1, it follows that $\tau = 1$, and hence also $R'^{(\tau\eta)} = R'^{(\eta)}$. As we have already seen above, the number of points $(Q'^{(\tau)}, R'^{(\tau\eta)})$ which are in $Q' \times \{\mathfrak{B}\}$ is independent of the way in which the specialization $(v) \to (v')$, with reference to k, is extended; this implies, in particular, that the number of points $R'^{(\eta)}$ which are in $\{\mathfrak{B}\}$ is for all the specializations of $(R^{(1)}, \cdots, R^{(g)})$ over $Q \to Q'$, with reference to k, the same. This completes the first part of the proof.

Next assume that there is only one point $R'^{(\eta)}$ in $\{\mathfrak{B}\}$. Then by what is proved above, it follows that there is only one point among the $(Q'^{(\tau)}, R'^{(\tau\eta)})$ which is in $Q' \times \{\mathfrak{B}\}$. Hence, by Theorem 2, $Q' \times \mathfrak{B}$ is linearly connected, and hence $\mathfrak{B}$ is linearly connected.

A direct consequence is the connectedness theorem mentioned in the introduction.

COROLLARY. *Let W be a subvariety of $U \times V$, where V is a projective variety. Suppose the projection of W on U is equal to U and let $[W:U] = 1$. Let Q' be simple on U. Then $W\{Q'\}$ is linearly connected.*

Note. This result is proved, except for the linearity, in [IV].

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REFERENCES.

[I] A. Weil, *Foundations of algebraic geometry*, Colloquium Publications, New York, 1946.

[II] ———, "Sur les critères d'équivalence en géométrie algébrique," *Mathematische Annalen*, vol. 128 (1954), pp. 95-127.

[III] O. Zariski, "Theory and applications of holomorphic functions on algebraic varieties over arbitrary groundfields," *Memoirs of the American Mathematical Society*, No. 5, 1951.

[IV] ———, *The connectedness theorem for birational transformations*, Princeton Series, No. 12 (A symposium in Honor of S. Lefschetz), 1957.

UNIQUENESS IN THE CAUCHY PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS.*[1]

By A. P. CALDERÓN.

1. Introduction. We begin with a few historical remarks. The classical theorem of Cauchy-Kowalewski asserts the existence of analytic solutions of analytic equations or systems of equations with analytic Cauchy data and their uniqueness in the class of analytic functions. In 1901 Holmgren [6] proved that the Cauchy problem has at most one continuous solution with continuous derivatives up to the order of the equations, provided that the latter are linear and have analytic coefficients. The condition on the coefficients is essential in Holmgren's argument and precludes the possibility of generalizing the result to the non-linear case. The first contribution in the direction of removing the assumption of analyticity was made in 1939 by T. Carleman [2], who proved the corresponding result in the case of two independent variables assuming that the characteristics of the equations are non-multiple. In view of the example of E. de Giorgi [3] (see also A. Plis [10]), who exhibited an example with multiple characteristics for which the Cauchy problem has more than one solution, the condition on the characteristics in Carleman's result no longer seems artificial, but, unfortunately, the restriction to two variables is indispensable to the method of proof. In the case of more than two variables, no essential progress was made until 1954 when C. Müller [8] studied a special second order equation. Later P. Hartman and A. Wintner [4] and E. Heinz [5] settled the case of quasi-linear second order equations with the Laplacian as principal part. (See also P. D. Lax [7]). Shortly thereafter, N. Aronszajn [1] extended this result to general second order elliptic case. For other results on quasilinear equations with principal parts with constant coefficients, see also L. Nirenberg [9]. In the present paper, we shall generalize Carleman's theorem to functions of any number of variables with two exceptions, the case of three variables for equations and three or four variables for systems. The presence of these

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exceptional cases is due to topological difficulties in our approach. Whether they are really exceptional or not, we do not know. Since the non-multiplicity of the characteristics plays an essential role in our argument and it does not seem to be possible to introduce additional independent variables in a differential problem without sometimes causing multiplicities to appear in the characteristics, we have been unable to reduce the exceptional cases to the other. In special instances, though, such as that of second and third order equations, our result holds for any number of variables.

Our method is based on a representation of linear differential operators by means of singular integral operators. More specifically, if Au is a linear differential operator of homogeneous order m defined on all Euclidean space and with bounded coefficients, then $Au = H\Lambda^m u$, where Λ is a square root of the negative of the Laplacian and H is a singular integral operator. For operators such as H, there is an approximate functional calculus which allows us to reduce the problem of uniqueness to a simple form. The first section of the paper will be devoted to discussing the operators H and the representation of A briefly described above. For the proof of the basic properties, we shall refer the reader to a paper by A. Zygmund and the author (see [11]).

2. Singular integral operators. We begin this section with an informal discussion. Consider the real k-dimensional Euclidean space E_k, complex valued functions $u(x)$ on E_k, and the operator

$$H(u) = a(x)u(x) + \int_{E_k} h(x, x-z)u(z)\,dz, \tag{1}$$

where $a(x)$ is complex, bounded and $h(x,z)$ is complex homogeneous of degree $-k$ with respect to z, that is, $h(x,\lambda z) = \lambda^{-k}h(x,z)$ for $\lambda > 0$, where λz is the point with coordinates $\lambda z_1, \lambda z_2, \cdots, \lambda z_k$, and $z = (z_1, \cdots, z_k)$, and $h(x,z)$ has mean value zero on the sphere $S_{k-1} = \{|z| = (z_1^2 + \cdots + z_k^2)^{\frac{1}{2}} = 1\}$ for each x. Under mild assumptions on $h(x,z)$, the integral above is meaningful as a principal value integral for $u(x)$, say, square integrable over E_k. If $a(x)$ and $h(x,z)$ are actually independent of x, then Hu can be easily computed by taking Fourier transforms. Specifically, if $\hat{u}(x)$ denotes the Fourier transform of $u(x)$,[1a] then

$$\widehat{Hu} = (a + \hat{h})\hat{u}, \tag{2}$$

[1a] In contrast with [11] we use here the Fourier integral with a minus sign in the exponent. This accounts for apparent discrepancies in sign.

where $\hat{h}$ is the Fourier transform of $h(x,z)=h(z)$, the Fourier integral being taken as a principal value integral. If $h(z)$ is continuous in $|z|>0$, then $\hat{h}(z)$ exists, is a homogeneous function of degree zero, that is, such that $\hat{h}(\lambda z)=\hat{h}(z)$ for every $\lambda>0$, and has mean value zero on $S_{k-1}=\{|z|=1\}$. Conversely, every sufficiently smooth function with these properties is the Fourier transform of some $h(z)$. The function $(a+\hat{h})$ in (2) is called the symbol of the operator H and we shall denote it by $\sigma(H)$. It is clear from (2) that operators with a and h independent of x can be multiplied and added by simply multiplying and adding their symbols.

Let now Au be a linear partial differential operator with constant coefficients of homogeneous order m. Then

$$\widehat{Au}=p_A(x)i^m\hat{u}(x), \tag{3}$$

where $p_A(x)$ is the characteristic form of the operator A. Let us rewrite the last identity as

$$\widehat{Au}=[p_A(x)|x|^{-m}i^m]|x|^m\,\hat{u}(x) \tag{4}$$

and define the operator Λ by

$$\widehat{\Lambda u}=|x|\,\hat{u}(x). \tag{5}$$

Then since the expression in square brackets in (4) is a homogeneous function of degree zero, it is the symbol of some singular integral operator H, and we obtain

$$\widehat{Au}=\sigma(H)|x|^m\hat{u}(x)=\sigma(H)\,(\Lambda^m u)^{\wedge}$$

and

$$Au=H\Lambda^m u, \tag{6}$$

where the symbol $\sigma(H)$ of H is given by $p_A(x)|x|^{-m}i^m$, $p_A(x)$ being the characteristic form of A. If A is the Laplacian, then

$$Au=\Delta u=-\Lambda^2 u,$$

which justifies the assertion that Λ is a square root of the negative of the Laplacian.

Suppose now that the coefficients of A are variable, that is,

$$Au=\sum_{\alpha} a_\alpha(x)D_\alpha u, \tag{7}$$

where $D_\alpha u$ runs over all partial derivatives of u of order m. Let us now represent each $D_\alpha u$ as in (16)

$$D_\alpha u=H_\alpha\Lambda^m u, \tag{8}$$

and substituting in (7), we obtain

$$Au = \sum_{\alpha} a_\alpha(x) H_\alpha \Lambda^m u = (\sum a_\alpha H_\alpha)\Lambda^m u = H\Lambda^m u, \tag{9}$$

where $Hu = \sum a_\alpha H_\alpha u = \sum a_\alpha(x)(H_\alpha u)$. Now as is easily seen, since H_α is of the form (1), so are $a_\alpha H_\alpha$ and $H = \sum a_\alpha H_\alpha$. Thus we have obtained the desired representation of A in the case of variable coefficients.

Let us revert to the operators of the form (1). In the general case, that is, when a and h actually depend on x, we shall define the symbol $\sigma(H)$ of the operator H by

$$\sigma(H) = a(x) + \hat{h}(x, z),$$

where $\hat{h}(x, z)$ is the Fourier transform of $h(x, z)$ with respect to z. The symbol is now a function of two arguments, $x, z \in E_k$, which is homogeneous of degree zero with respect to z. One verifies that the symbol of the operator H in (9) is precisely $\sum a_\alpha(x) z^\alpha \mid z \mid^{-m} i^m$, where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$, and $\sum a_\alpha(x) z^\alpha$ is the characteristic form of the operator A.

The operators of the form (1) do not form an algebra under ordinary composition but it is possible to introduce a product $H_1 \circ H_2$ by means of the formula $\sigma(H_1 \circ H_2) = \sigma(H_1)\sigma(H_2)$. Under mild restrictions on the operators H, this product is always defined, and with this multiplication the class of singular integral operators becomes a commutative algebra which is isomorphic to the function algebra of their symbols. The interest of this kind of multiplication lies in the fact that it is a fairly satisfactory substitute for ordinary composition of operators which, incidentally, is non-commutative. More specifically, the difference between $H_1 \circ H_2$ and $H_1 H_2$ has the following property. Our operators H are bounded on $L^2(E_k)$ and Λ is densely defined in $L^2(E_k)$ and unbounded, but $H_1 \circ H_2 - H_1 H_2$ multiplied on the right or the left by Λ is bounded. Thus, modulo operators with the property of remaining bounded after multiplication by Λ, ordinary composition is equivalent to the formally simpler multiplication introduced above.

We now proceed to formulate precisely the properties of singular integral operators which we shall need. For their proof we refer the reader to [11].[12] By C_β, $1 \leqq \beta \leqq 2$, we shall mean the class of bounded complex valued functions in C_1 with bounded first order derivatives satisfying a uniform Hölder condition of order $\beta - 1$. If $0 \leqq \beta < 1$, then C_β will stand for the class of bounded continuous functions satisfying a uniform Hölder condition of order β. By C_β^∞, we shall mean the class of functions depending on two or more arguments which are in C^∞ with respect to the last argument and whose

derivatives of all orders with respect to the coordinates of this are in C_β with respect to all variables. By L^2, we shall mean the class of functions on E_k with integrable square, and L_1^2 will stand for the class of functions in L^2 with first order derivatives (in the sense of Schwartz) in L^2. We shall say that the operator (1) is of type C_β^∞ if $a(x) \in C_\beta$ and $h(x,z) \in C_\beta^\infty$ in $|z| \geqq 1$. The letter c will stand for a constant not necessarily the same in each occurrence.

THEOREM 1. *Let $\beta \geqq 0$ be given. Consider the class of singular integral operators H of type C_β^∞. Then Hu is well defined as a principal value integral for $u \in L^2$ and Hu is in L^2. Further, H is a bounded operator on L^2. There exists a one-to-one linear mapping σ of the class of operators of type C_β^∞ onto the class of functions $F(x,z)$, $x, z \in E_k$ homogeneous of degree zero with respect to z and in C_β^∞ in $|z| \geqq 1$, such that, if N_H is the least upper bound for the absolute value of $F(x,z) = \sigma(H)$ (see footnote 2) and its derivatives with respect to coordinates of z of order $2k$ evaluated in $|z| \geqq 1$, then*

$$\| H \| \leqq c N_H, \tag{10}$$

where $\| H \|$ is the norm of H as an operator on L^2 and c is a constant depending only on k.

THEOREM 2. *If $u(x) \in L_1^2$ and $\hat{u}(x)$ denotes the Fourier transform of $u(x)$, then $\hat{u}(x)|x|$ belongs to L^2. Thus the operator Λ defined by $\widehat{\Lambda u} = \hat{u}(x)|x|$ has domain L_1^2, range in L^2 (see footnote 3) and is symmetric.*

THEOREM 3. *Let H be a singular integral operator of type C_β^∞ with $1 < \beta \leqq 2$. Let M_H denote the least upper bound for the absolute values of $\sigma(H) = F(x,z)$, its derivatives with respect to coordinates of z of order $2k$, the first order derivatives of these and of $F(x,z)$ with respect to coordinates of x, and the Hölder constants of the latter in $|z| \geqq 1$. Let H^* denote the adjoint of H and let $H^\#$, $H_1 \circ H_2$ be defined by*

$$\sigma(H^\#) = \bar{\sigma}(H), \qquad \sigma(H_1 \circ H_2) = \sigma(H_1)\sigma(H_2). \tag{11}$$

Then $H^\#$ and $H_1 \circ H_2$ are singular integral operators of type C_β^∞. Further, if H is of type C_β^∞ and $u \in L_1^2$, then Hu and H^*u are in L_1^2, so that the operators ΛH, $H\Lambda$, $H^*\Lambda$, ΛH^* are defined on L_1^2. Further, if $u \in L_1^2$, then

[2] The function $\sigma(H)$ here is the symbol of H.

[3] The definition of the operator Λ given here is different from the one given in [11]. They are, nevertheless, readily seen to be equivalent.

$$
\begin{gathered}
\|(\Lambda H - H\Lambda)u\| \leqq cM_H \|u\|, \qquad \|\Lambda(H^* - H^\#)u\| \leqq cM_H \|u\|, \\
\|(H^* - H^\#)\Lambda u\| \leqq cM_H \|u\|, \\
\|\Lambda(H_1 \circ H_2 - H_1H_2)u\| \leqq cM_{H_1}M_{H_2} \|u\|, \\
\|(H_1 \circ H_2 - H_1H_2)\Lambda u\| \leqq cM_{H_1}M_{H_2} \|u\|,
\end{gathered} \tag{12}
$$

where c is a constant depending on k only and the norms are L^2-norms. Consequently, the operators in (12) *can be extended continuously to bounded operators on L^2.*

THEOREM 4. *If H_1 and H_2 are in C_β^∞, $\beta \geqq 0$ and the functions $\sigma(H_1) = F_1(x, z)$, $\sigma(H_2) = F_2(x, z)$ are independent of x, then $H_1 \circ H_2 = H_1H_2$, $H_1^\# = H_1^*$ and H_1 has an inverse if and only if $\sigma(H_1)$ does not vanish. Further, H_1 takes L_1^2 into L_1^2 and $H_1\Lambda = \Lambda H_1$.*

THEOREM 5. *Let $A = \sum_\alpha a_\alpha(x) D_\alpha$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k = m$, and $D_\alpha = \partial^{\alpha_1+\alpha_2+\cdots+\alpha_k}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2} \cdots \partial x_k^{\alpha_k}$ be a linear homogeneous differential operator of homogeneous order m with coefficients $a_\alpha(x)$ in C_β, $\beta \geqq 0$. If $u(x)$ is in L^2 and has derivatives of orders less than or equal to m in L^2, then $\Lambda^m u$ is defined and $Au = H\Lambda^m u$, where H is a singular integral operator of type C_β^∞ and*

$$
\sigma(H) = i^m \sum a_\alpha(x) z^\alpha |z|^{-m}, \tag{13}
$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$. In particular, if $u \in L_1^2$, then

$$
\partial u/\partial x_j = iR_j\Lambda u = i\Lambda R_j u, \tag{14}
$$

where $\sigma(R_j) = z_j |z|^{-1}$. Conversely, if $u \in L_1^2$, then

$$
i\Lambda u = \sum_{j=1}^{k} R_j \partial u/\partial x_j = \sum_{j=1}^{k} \partial R_j u/\partial x_j. \tag{15}
$$

In what follows, we shall consider singular integral operators depending on a parameter t and function $u(t)$ depending on t and with values in L^2. We shall say that such a function is continuous if $\|u(t) - u(t_0)\|$ tends to zero as t tends to t_0 for all t_0. We shall say that $u(t)$ is differentiable if there exists a function $du/dt = v(t)$ of the same type such that

$$
\|(u(t) - u(t_0))/(t - t_0) - v(t_0)\|
$$

tends to zero as t tends to t_0, for all t_0.

Our results on uniqueness are essentially applications of the following

LEMMA 1. *Let $u(t)$, $0 \leqq t \leqq h$, have values in L_1^2, be continuously*

differentiable and be such that $\Lambda u(t)$ is continuous. Let $P(t)$ and $Q(t)$ be singular integral operators such that $\sigma(P) = F_1(t,x,z)$ and $\sigma(Q) = F_2(t,x,z)$ are real and in C_β^∞, $\beta > 1$ in $|z| \geqq 1$.[4] *Assume that $P(t)$ has a two sided inverse for each t or else vanishes identically, and let $\phi_n(t) = (t+1/n)^{-n}$. Then if $u(0) = 0$ and*

$$\int_0^h \phi_n^2 \| du/dt + (P+iQ)\Lambda u \|^2 \, dt \leqq c \int_0^h \phi_n^2 \| u \|^2 \, dt \tag{16}$$

for some constant c and arbitrarily large n, $u(t) = 0$ in a neighborhood of $t = 0$.

Proof. We will first derive inequality (23) and then show that (16) and (23) are incompatible for sufficiently large n, unless $u(t) = 0$ in a neighborhood of $t = 0$. We shall write u' for du/dt etc. and shall omit displaying supscript n of ϕ_n.

We have

$$\phi[u' + (P+iQ)\Lambda u] = [(\phi u)' + iQ\Lambda\phi u] + [P\Lambda\phi u - \phi' u].$$

Squaring, we obtain

$$\begin{aligned} \phi^2 \| u' + (P+iQ)\Lambda u \|^2 &= \| (\phi u)' + iQ\Lambda\phi u \|^2 + \| P\Lambda\phi u - \phi' u \|^2 \\ &- [(\phi u)', \phi' u] - [\phi' u, (\phi u)'] + [(\phi u)', P\Lambda\phi u] + [P\Lambda\phi u, (\phi u)'] \\ &+ i[Q\Lambda\phi u, P\Lambda\phi u] - i[P\Lambda\phi u, Q\Lambda\phi u] - i[Q\Lambda\phi u, \phi' u] + i[\phi' u, Q\Lambda\phi u], \end{aligned} \tag{17}$$

where the brackets denote inner products of the functions involved. Now we integrate between 0 and h and estimate the integrals of the last eight terms. First, we have

$$[(\phi u)', \phi' u] + [\phi' u, (\phi u)'] = 2\phi'^2 \| u \|^2 + \phi\phi'[(u', u) + (u, u')],$$

and integrating between 0 and h we obtain

$$\begin{aligned} &\int_0^h \{[(\phi u)', \phi' u] + [\phi' u, (\phi u)']\} dt \\ &\quad = 2\int_0^h \phi'^2 \| u \|^2 \, dt - \int_0^h (\phi\phi'' + \phi'^2) \| u \|^2 \, dt + \phi(h)\phi'(h) \| u(h) \|^2. \end{aligned}$$

Now one verifies readily that

$$\phi\phi'' + \phi'^2 = 2\phi'^2 + \phi'^2/n$$

---

[4] Note that both F_1 and F_2 are assumed to be in C_β^∞ as functions of all their variables, t included.

and that $\phi(h)\phi'(h)\| u \|^2 \leqq 0$. Consequently, it follows that

$$\int_0^h [(\phi u)', \phi' u]dt + [\phi' u, (\phi u)']dt \leqq -n^{-1}\int_0^h \phi'^2 \| u \|^2 dt. \tag{18}$$

Next we estimate the integral of the terms $[(\phi u)', P\Lambda\phi u] + [P\Lambda\phi u, (\phi u)']$. Let us write $v = \phi u$ and P' for the operator defined by $\sigma(P') = (\partial/\partial t)\sigma(P)$ (i.e., P' is the derivative of $P(t)$). Since $\sigma(P) \in C_\beta^\infty$, $\beta > 1$ in $|z| \geqq 1$, using (10), one verifies readily that Pv is differentiable and that $(Pv)' = P'v + Pv'$. Since v is in L_1^2, it follows from Theorem 3 that Pv is in L_1^2 and ΛPv is defined. Further, since $\Lambda v = \phi\Lambda u$ is continuous, writing w for Pv, we have

$$\begin{aligned}&\{[v(t+s), \Lambda w(t+s)] - [v(t), \Lambda w(t)]\}/s \\ &= [\Lambda v(t+s), (w(t+s) - w(t))/s] + [(v(t+s) - v(t))/s, \Lambda w(t)],\end{aligned}$$

and as s tends to zero, the right hand side tends towards $(\Lambda v, w') + (v', \Lambda w)$; that is, $(v, \Lambda w)$ is differentiable and its derivative is equal to the expression above. More explicitly, we have

$$(\phi u, \Lambda P\phi u)' = [\Lambda\phi u, P(\phi u)'] + [\Lambda\phi u, P'\phi u] + [(\phi u)', \Lambda P\phi u].$$

Consequently, we can write

$$\begin{aligned}&[(\phi u)', P\Lambda\phi u] + [P\Lambda\phi u, (\phi u)'] \\ &= [\phi u, \Lambda P\phi u]' - [\Lambda\phi u, P'\phi u] \\ &\quad + [(P - P^*)\Lambda\phi u, (\phi u)'] + [(\phi u)', (P\Lambda - \Lambda P)\phi u].\end{aligned}$$

Since $P = P^\#$, according to (12), the operators $(P - P^*)\Lambda$ and $P\Lambda - \Lambda P$ are bounded and their norms, as functions of t, are bounded. Also, according to (10), the norm of P' is bounded. So if we replace $(\phi u)'$ by $\phi' u + \phi u'$ above and integrate, we obtain

$$\begin{aligned}&\int_0^h [(\phi u)', P\Lambda\phi u]dt + [P\Lambda\phi u, (\phi u)']dt \\ &\geqq \phi^2(h)[u(h), \Lambda P(h)u(h)] - c\int_0^h [\phi^2 \| \Lambda u \| \| u \| \\ &\quad + |\phi\phi'| \| u \|^2 + \phi^2 \| u \| \| u' \|]dt,\end{aligned} \tag{19}$$

where c is a constant independent of n. Next we have

$$\begin{aligned}&[Q\Lambda\phi u, P\Lambda\phi u] - [P\Lambda\phi u, Q\Lambda\phi u] = [(P^*Q - Q^*P)\Lambda\phi u, \Lambda\phi u] \\ &= [(P^\#Q - Q^\#P)\Lambda\phi u, \Lambda\phi u] + [\Lambda(P^* - P^\#)Q\Lambda\phi u, \phi u] \\ &\quad - [\Lambda(Q^* - Q^\#)P\Lambda\phi u, \phi u].\end{aligned} \tag{20}$$

Now, according to (12), the operators $\Lambda(P^* - P^\#)$ and $\Lambda(Q^* - Q^\#)$ are bounded and their norms, as functions of t, are bounded. Further, since $P = P^\#$ and $Q = Q^\#$ and $P \circ Q = Q \circ P$ we have

$$(P^\# Q - Q^\# P)\Lambda = (PQ - P \circ Q)\Lambda - (QP - Q \circ P)\Lambda,$$

and according to (12), the operators on the right are bounded and their norms, as function of t, are bounded. Consequently, integrating (20), we obtain

$$(21) \qquad \int_0^h \{i[Q\Lambda\phi u, P\Lambda\phi u] - i[P\Lambda\phi u, Q\Lambda\phi u]\}dt \geqq -c\int_0^h \phi^2 \| u \| \| \Lambda u \| dt.$$

Finally, $i[\phi' u, Q\Lambda\phi u] - i[Q\Lambda\phi u, \phi' u] = i\phi\phi'[(\Lambda Q^* - Q\Lambda)u, u]$, and since $Q = Q^\#$, it follows again that $\Lambda Q^* - Q\Lambda$ is a bounded operator and has bounded norm. Thus, integrating, we obtain

$$(22) \qquad \int_0^h \{i[\phi' u, Q\Lambda\phi u] - i[Q\Lambda\phi u, \phi' u]\}dt \geqq -c\int_0^h |\phi\phi'| \| u \|^2 dt.$$

Now we integrate (17) and using (18), (19), (21), and (22) and observing that $\phi(h) \leqq h^{-n}$, we obtain

$$(23) \qquad \int_0^h \phi^2 \| u' + (P + iQ)\Lambda u \|^2 dt \geqq \int_0^h \| P\Lambda\phi u - \phi' u \|^2 dt$$
$$+ n^{-1}\int_0^h \phi'^2 \| u \|^2 dt - c\int_0^h [|\phi\phi'| \| u \|^2 + \phi^2 \| u \| \| \Lambda u \|$$
$$+ \phi^2 \| u \| \| u' \|] dt - ch^{-2n},$$

where c is a constant independent of n. Now we will show that this last inequality is incompatible with (16) for sufficiently large n, unless $u(t) = 0$ in a neighborhood of $t = 0$. Assume first that $P(t)$ does not vanish identically and set

$$(24) \qquad \int_0^h \phi'^2 \| u \|^2 dt = I^2; \qquad \int_0^h \phi^2 \| P\Lambda u \|^2 dt = \rho^2 I^2.$$

Then one verifies readily that, unless $u(t) = 0$ in a neighborhood of $t = 0$, we have

$$(25) \qquad \int_0^h \phi^2 \| u \|^2 dt = o(I^2/n^2)$$

and $h^{2n} I^2/n^2 \to \infty$ as n tends to infinity, which implies that

$$(26) \qquad h^{-2n} = o(I^2/n^2).$$

Since $P(t)$ has a two sided inverse $P(t)^{-1}$ for each t and is continuous in the norm topology for operators on L^2 (this is an immediate consequence of

(10) and our assumptions on $P(t)$), it follows that $P(t)^{-1}$ has bounded norm and, consequently, $\| \Lambda u \| \leqq c \| P\Lambda u \|$, where c is independent of t, which in turn implies that

$$\int_0^h \phi^2 \| \Lambda u \|^2 \, dt = O(\rho^2 I^2). \tag{27}$$

Since P and Q are bounded operators and have bounded norms, for those n for which (16) holds, we have from (16), (25) and (27)

$$\int_0^h \phi^2 \| u' \|^2 \, dt = O(\rho I + I/n)^2 \tag{28}$$

Finally, for the same n from (24), (25), (27), and (28), we obtain

$$\int_0^h \| P\Lambda\phi u - \phi' u \|^2 \, dt \geqq (\rho I - I)^2, \int_0^h | \phi\phi' | \| u \|^2 \, dt = o(I^2/n), \tag{29}$$

$$\int_0^h \phi^2 \| u \| \, \| \Lambda u \| \, dt = o(\rho I^2/n), \int_0^h \phi^2 \| u \| \, \| u' \| \, dt = o(\rho I^2/n + I^2/n^2).$$

Now the right-hand side of (16) is larger than or equal to the right-hand side of (23) for all n for which (16) holds. Hence, for such n, from the estimates (24) to (29), we obtain

$$o(I^2/n^2) \geqq (\rho - 1)^2 I^2 + I^2/n - o(I^2/n + \rho I^2/n + I^2/n^2) - o(I^2/n^2).$$

Now, neglecting the higher order terms $o(I^2/n)$ and $o(I^2/n^2)$ multiplying by n and dividing by ρI^2 we get

$$o(1) \geqq (n(\rho - 1)^2 + 1)/\rho \geqq ((\rho - 1)^2 + 1)/\rho > \tfrac{1}{2}$$

which is, of course, impossible for sufficiently large n.

If $P(t) = 0$ identically, then (23) reduces to

$$\int_0^h \phi^2 \| u' + iQ\Lambda u \|^2 \, dt \geqq (1 + 1/n) \int_0^h \phi'^2 \| u \|^2 - c \int_0^h | \phi\phi' | \| u \|^2 \, dt$$

and this combined with (16), (24) and (25) gives

$$o(I^2/n^2) \geqq (1 + 1/n) I^2 - o(I^2/n)$$

which again is readily seen to be impossible for sufficiently large n. Lemma 1 is thus established.

LEMMA 2. *Let $u(t)$, $0 \leqq t \leqq h$ have values in L^2 and be continuously differentiable. Then*

$$\int_0^h \phi_n^2 \| du/dt \|^2 \, dt \geqq \int_0^h \phi_n'^2 \| u \|^2 \, dt \geqq n^2 (1 + h)^{-2} \int_0^h \phi_n^2 \| u \|^2 \, dt. \tag{30}$$

Proof. To obtain the first inequality, we merely set $P = Q = 0$ in (17), integrate and use (18). The second inequality follows from the fact that $\phi_n'^2 \geqq n^2(1+h)^{-2}\phi_n^2$ in $0 \leqq t \leqq h$.

3. Uniqueness in the case of a single linear equation. Our problem consists in showing that if $u \in C_m$ is a solution of a linear homogeneous partial differential equation $A(u) = 0$ of order m, and u and its derivatives of order less than or equal to m vanish on a non-characteristic manifold M, then u must vanish in a neighborhood of M. The problem is local, and we need only consider what happens in an arbitrary neighborhood of a given point of M.

We shall consider the function u only on one side of M, and, for this purpose, it will be convenient to replace u by zero on the other, which will neither disturb the fact that u is a solution of $A(u) = 0$ nor destroy the assumed continuity properties of u. Further, and without loss of generality, we shall assume that M is of class C_m. The differential operator A will be assumed to have non-multiple characteristics, coefficients of highest order derivatives in C_β, $\beta > 1$, and remaining ones measurable and bounded. We shall first bring our differential equation to a suitable form and then replace by it an inequality involving singular operators to which, after some manipulation, we shall apply Lemma 1.

We shall denote by $k+1$ the number of independent variables. By introducing suitable coordinates $t, x_1, x_2, \cdots, x_k$, we can assume that M coincides with the paraboloid $t = |x|^2 = [\sum_1^k x_j^2]^{\frac{1}{2}}$ and that $u = 0$ in $t \leqq |x|^2$, and we shall show that $u = 0$ for $t \leqq h$, for some $h > 0$.

Since $u = 0$ in $t \leqq |x|^2$, we may freely modify the coefficients of A there, and, in fact, we shall replace A by

$$A' = g_0 A + (1 - g_0) A_0, \tag{31}$$

where g_0 is a function in C_∞ such that, $0 \leqq g_0 \leqq 1$, $g_0 = 1$ in $t \geqq |x|^2$ and $g_0 = 0$ in $t \leqq |x|^2 - \delta$, $\delta > 0$, and A_0 is the operator obtained from A by replacing its coefficients by their values at $t = x = 0$. We note that A' coincides with A at $t = x = 0$ regardless of the value of δ. By taking δ sufficiently small and by restricting our attention to a sufficiently small interval $0 \leqq t \leqq h$, we may assume that the oscillation of the coefficients of A' is less than any preassigned number ϵ. Since $t = |x|^2$ is non-characteristic, the hyperplane $t = 0$ is non-characteristic at $t = x = 0$. Consequently, we may assume that the coefficient of $\partial u^m / \partial t^m$ does not vanish in

a neighborhood of $t=x=0$. Hence, the coefficient of $\partial^m u/\partial t^m$ in A' does not vanish for $0 \leqq t \leqq h$, we may write our equation in the form

$$\partial^m u/\partial t^m + \sum_{j=1}^{m} A_j \partial u^{m-j}/\partial t^{m-j} + B(u) = 0, \tag{32}$$

where A_j is a linear partial differential operator of homogeneous order j involving only differentiations with respect to $x_1, \cdots x_k$, and $B(u)$ is linear homogeneous and of order less than m. Our previous considerations make it clear that the characteristics of (32) can be assumed to be nowhere multiple and the oscillation of the coefficients of the A_j arbitrarily small.

We shall regard the function $u(t,x)$ in (32) as a function $u(t)$ with values in $L^2(E_k)$. Since $u(t,x)$ has continuous derivatives up to order m, and vanishes for x outside a fixed bounded set, the functions

$$v_j(t) = (i\Lambda)^{m-j} d^{j-1} u/dt^{j-1}, \qquad j = 1, 2, \cdots, m, \tag{33}$$

have values in L_1^2, are continuously differentiable and the Λv_j are continuous.

Now we represent the operators A_j in (32) as in Theorem 5; that is, we set $A_j = i^j H_j \Lambda^j$, where H_j is the singular integral operator defined by

$$\sigma(H_j) = p_j(t,x,z) \mid z \mid^{-j},$$

$p_j(t,x,z)$ being the characteristic form of A_j. Thus (32) can be rewritten as

$$dv_m/dt + \sum_{1}^{m} H_j i\Lambda v_{m-j+1} = -B(u), \qquad dv_j/dt - i\Lambda v_{j+1} = 0, \tag{34}$$

the last equation being an immediate consequence of (33).

Now we shall write these equations in matrix form. For this purpose we shall consider $m \times m$ matrices of singular integral operators and regard them as operators acting on m-vectors with functions in $L^2(E_k)$ as components. By the norm of such a vector, we shall mean the square root of the sum of the squares of the norms of its components. The meaning of the norm of such a matrix operator is clear, and the way in which the properties of singular integral operators extend to matrix operators is also apparent. Thus, if we write $v(t)$ for the vector of components $v_1(t), v_2(t), \cdots, v_m(t)$, and $w(t)$ for $[0, 0, \cdots, -B(u)]$, and if $\mathfrak{A}(t)$ denotes the matrix operator of entries $H_{ij} = -I$ (I being the identity operator) if $j = l+1$, $H_{mj} = H_{m-j+1}$, and remaining ones equal to zero, we may write (34) as

$$dv/dt + \mathfrak{A}(t) i\Lambda v = w(t). \tag{35}$$

Now we will estimate $\| w \|^2$ in terms of $\| v \|^2$. Since B is a sum of linear

homogeneous partial differential operators, each of homogeneous order, applied to the derivatives $\partial^j u/\partial t^j$, representing each of these operators as in Theorem 5, we obtain

$$\| Bu \|^2 \leqq c \sum_{l+j \leqq m-1} \| \Lambda^l d^j u/dt^j \|^2 = c \sum_{l+j \leqq m-1} \| d^j \Lambda^l u/dt^j \|^2,$$

and a repeated application of Lemma 2 to each of the terms of the last sum gives

$$\int_0^h \phi_n^2 \| Bu \|^2 \leqq c \sum_j \int_0^h \phi_n^2 \| d^{m-j-1} \Lambda^j u/dt^{m-j-1} \|^2 \, dt,$$

or, according to (33), $\int_0^h \phi_n^2 \| w \|^2 \, dt \leqq c \int_0^h \phi_n^2 \| v \|^2 \, dt$. From this and (35), it follows that

$$\int_0^h \phi_n^2 \| dv/dt + \mathcal{H}(t) i\Lambda v \|^2 \, dt \leqq c \int_0^h \phi_n^2 \| v \|^2 \, dt. \tag{36}$$

In order to apply Lemma 2, we shall replace this inequality by a similar one involving a diagonal matrix instead of $\mathcal{H}(t)$.

Consider the matrix $\sigma(\mathcal{H})$ of entries $\sigma(H_{ij})$ and the characteristic polynomial $\det[\lambda I + \sigma(\mathcal{H})]$ of $-\sigma(\mathcal{H})$. Then a simple computation gives

$$\begin{aligned} \det[\lambda I + \sigma(\mathcal{H})] &= \lambda^m + p_1(t,x,z) \mid z \mid^{-1} \lambda^{m-1} + \cdots + p_{m-1}(t,x,z) \mid z \mid^{-m+1} \lambda \\ &\quad + p_m(t,x,z) \mid z \mid^{-m}, \end{aligned}$$

where $p_j(t,x,z)$ is the characteristic form of the operator A_j in (32). But as is readily seen, this function coincides on $\mid z \mid = 1$ with the characteristic form of the equation (32) which was assumed to be non-multiple. Consequently, for each (t,x,z), $\mid z \mid > 0$, the matrix $\sigma(\mathcal{H})$ has distinct characteristic values $\lambda_1, \lambda_2, \cdots, \lambda_m$.

Let us postpone the discussion of the cases $k = 1$ and $k = 2$, and assume temporarily that $k \geqq 3$. If $k \geqq 3$, the set of points (t,x,z), $\mid z \mid > 0$, is simply connected and the principle of monodromy implies that there exist single-value continuous functions $\lambda_1(t,x,z), \lambda_2(t,x,z), \cdots, \lambda_m(t,x,z)$ coinciding at each point with a characteristic value of $\sigma(\mathcal{H})$. Further, since the entries of $\sigma(\mathcal{H})$ are real, the $\lambda_j(t,x,z)$ which are complex can be paired in complex conjugate pairs. This implies that the imaginary part of a complex $\lambda_j(t,x,z)$ cannot vanish without the characteristics of (32) becoming multiple at a point. Hence the functions $\lambda_j(t,x,z)$ are either real or else their imaginary parts do not vanish. We note that since the entries of $\sigma(\mathcal{H})$ are functions of (t,x,z) in C_β^∞, homogeneous of degree zero with respect

to z, the same holds for the $\lambda_j(t,x,z)$. Further, the oscillation of the entries of $\sigma(\mathcal{H})$ can be assumed to be arbitrarily small for each fixed z, and consequently, the same holds for the $\lambda_j(t,x,z)$.

Define now the matrix $N_1(t,x,z)$ by $N_1(t,x,z) = (a_{ij}(t,x,z))$, where $a_{ij} = \lambda_j(t,x,z)^{i-1}$. Then, since the λ_j are all distinct, N_1 is nonsingular and has an inverse. Set $N = N_1^{-1}$ and denote by D the diagonal matrix with entries $-\lambda_j(t,x,z)$ in the main diagonal. Then $\sigma(\mathcal{H})N_1 = N_1 D$, and consequently, $N\sigma(\mathcal{H}) = DN$. Let $N_0 = N(0,0,z)$ and define the matrix singular integral operators, $\mathcal{N}$, $\mathcal{N}_0$ and $\mathcal{D}$, by $\sigma(\mathcal{N}) = N$, $\sigma(\mathcal{N}_0) = N_0$, $\sigma(\mathcal{D}) = D$. Since N_0 is nonsingular, it follows from Theorem 4 that the operator $\mathcal{N}_0$ is invertible. Since the entries of $\sigma(\mathcal{N} - \mathcal{N}_0)$ and their partial derivatives with respect to coordinates of z of order $2k$ can be assumed to be arbitrarily small,[5] it follows from (10) that the norm of $\mathcal{N} - \mathcal{N}_0$ can be assumed to be arbitrarily small, and since $\mathcal{N}_0$ has an inverse $\mathcal{N} = \mathcal{N}(t)$ has an inverse with bounded norm. Consequently,

$$\int_0^h \phi_n^2 \| \mathcal{N} dv/dt + \mathcal{N}\mathcal{H} i\Lambda v \|^2 dt \leqq c \int_0^h \phi_n^2 \| dv/dt + \mathcal{H} i\Lambda v \|^2 dt$$

and $\int_0^h \phi_n^2 \| v \|^2 dt \leqq c \int_0^h \phi_n^2 \| \mathcal{N} v \|^2 dt$, and from (36), we obtain

$$\int_0^h \phi_n^2 \| \mathcal{N} dv/dt + \mathcal{N}\mathcal{H} i\Lambda v \|^2 dt \leqq c \int_0^h \phi_n^2 \| \mathcal{N} v \|^2 dt. \tag{37}$$

Now $\sigma(\mathcal{N})$ is a function of (t,x,z) of C_β^∞, $\beta > 1$ in $|z| \geqq 1$. Consequently, if we define $d\mathcal{N}/dt$ by $\sigma(d\mathcal{N}/dt) = \partial\sigma(\mathcal{N})/\partial t$, then $d\mathcal{N}/dt$ has bounded norm and

$$\mathcal{N} dv/dt = d(\mathcal{N} v)/dt - (d\mathcal{N}/dt)v = d(\mathcal{N} v)/dt + v', \tag{38}$$

where $\| v' \| \leqq c \| v \|$, and consequently, also $\| v' \| \leqq c \| \mathcal{N} v \|$. Further, since

$$\sigma(\mathcal{N})\sigma(\mathcal{H}) = N\sigma(\mathcal{H}) = DN = \sigma(\mathcal{D})\sigma(\mathcal{N}),$$

we have $\mathcal{N} \circ \mathcal{H} = \mathcal{D} \circ \mathcal{N}$ and

$$\mathcal{N}\mathcal{H} i\Lambda = \mathcal{D} i\Lambda \mathcal{N} + \mathcal{D} i(\mathcal{N}\Lambda - \Lambda\mathcal{N}) + (\mathcal{D} \circ \mathcal{N} - \mathcal{D}\mathcal{N}) i\Lambda + (\mathcal{N}\mathcal{H} - \mathcal{N} \circ \mathcal{H}) i\Lambda.$$

But according to Theorem 3, the last three terms on the right represent bounded operators, and consequently, we may write

$$\mathcal{N}\mathcal{H} i\Lambda v = \mathcal{D} i\Lambda \mathcal{N} v + v'', \tag{39}$$

[5] Note that the oscillations of the entries of this matrix and of its derivatives with respect to coordinates of z can be estimated in terms of the oscillation of the coefficients of the equation (32).

where $\| v'' \| \leqq c \| v \|$, and therefore, also $\| v'' \| \leqq c \| \mathcal{N} v \|$. Substituting (38) and (39) in (37), we obtain

$$\int_0^h \phi_n^2 \| d(\mathcal{N} v)/dt + \mathcal{D} i\Lambda(\mathcal{N} v) \|^2 dt \leqq c \int_0^h \phi_n^2 \| \mathcal{N} v \|^2 dt$$

$$+ c \int_0^h \phi_n^2 \| v' \|^2 dt + c \int_0^h \phi_n^2 \| v'' \|^2 dt \leqq c \int_0^h \phi_n^2 \| \mathcal{N} v \|^2 dt. \tag{40}$$

Let now $w_1, w_2, \cdots, w_m$ be the components of $\mathcal{N} v$, and $P_1 + iQ_1$, $P_2 + iQ_2$, $\cdots, P_m + iQ_m$ the entries in the main diagonal if $i\mathcal{D}$, where $-\sigma(Q_j)$ and $\sigma(P_j)$ are, respectively, equal to the real and imaginary part of $\lambda_j(t, x, z)$. Since the imaginary part of $\lambda_j(t, x, z)$ is identically zero or else does not vanish, and since the oscillation of $\lambda_j(0, 0, z) - \lambda_j(t, x, z)$ and its derivatives with respect to coordinates of z of order $2k$ can be assumed to be arbitrarily small, it follows from (10) and Theorem 4, that either P_j vanishes identically or else has an inverse. Thus (40) can be written as

$$\int_0^h \phi_n^2 \| dw_j/dt + (P_j + iQ_j)\Lambda w_j \|^2 dt \leqq c \sum_{l=1}^m \int \phi_n^2 \| w_l \|^2 dt, \tag{41}$$

$$j = 1, 2, \cdots, m.$$

Let now j, $1 \leqq j \leqq m$, be an index such that

$$\int_0^h \phi_n^2 \| w_j \|^2 dt \geqq 1/m \int_0^h \phi_n^2 \sum_{l=1}^m \| w_l \|^2 dt \tag{42}$$

for infinitely many n. Then for such an index, (41) yields

$$\int_0^h \phi_n^2 \| dw_j/dt + (P_j + iQ_j)\Lambda w_j \|^2 dt \leqq c \int_0^h \phi_n^2 \| w_j \|^2 dt$$

for infinitely many n. Now, as is readily verified, the remaining assumptions of Lemma 1 are satisfied, and thus $w_j(t) = 0$ in a neighborhood of $t = 0$. But then (42) implies that all the other components of $w = \mathcal{N} v$ must vanish in the same interval. Consequently, $\mathcal{N} v = 0$ for t sufficiently small, whence $v = 0$ and $u = 0$ for the same values of t.

Thus uniqueness is established, provided $k \geqq 3$.

To complete the proof in the cases $k = 1$ and $k = 2$, we still have to construct the functions $\lambda_j(t, x, z)$ coinciding at each point with a characteristic value of $\sigma(\mathcal{A})$. If $k = 1$, the set of points (t, x, z), $|z| > 0$, has two simply connected components. On each of these, we can construct the functions $\lambda_j(t, x, z)$ as we did in the case $k \geqq 3$, and they will turn out to be either real or else have a non-vanishing imaginary part. Then we pair the functions on each component in such a way that $\lambda_j(t, x, z) = -\lambda_j(t, x, -z)$

and this will yield the functions $\lambda_j(t,x,z)$ over the whole set (t,x,z), $|z|>0$, with the desired property of either being real or else having a non-vanishing imaginary part.

If $k=2$, the set of points (t,x,z), $|z|>0$, is no longer simply connected; i.e., the principle of monodromy does not hold, and indeed, it is possible to construct a differential operator for which no continuous single valued functions $\lambda_j(t,x,z)$ exist. For such an operator, our argument can be carried no further and the question of uniqueness remains open. Yet, in a special case which is certainly not devoid of interest, the functions $\lambda_j(t,x,z)$ still exist. This is when there exist at most two complex characteristic values of $\sigma(\mathcal{A})$ at each point (t,x,z), $|z|>0$. To show the existence of the $\lambda_j(t,x,z)$, we first construct the functions $\lambda_j(t,x,y)$ on the universal covering space of the set (t,x,z), $|z|>0$. The same argument used previously shows that either $\lambda_j(t,x,y)$ is real at all points or else its imaginary part does not vanish. Let us denote by λ_1 and λ_2 the two complex λ_j and assume that at (t_0,x_0,y_0), $\mathrm{Im}[\lambda_1]>0, \mathrm{Im}[\lambda_2]<0, \lambda_3<\lambda_4<\cdots<\lambda_m$. Since the λ_j are distinct at each point, these inequalities hold everywhere on the covering space. Now if (t_1,x_1,y_1) and (t_2,x_2,y_2) are two points above the same point (t,x,z), then the sets of values of the λ_j at these points coincide. Since the inequalities above hold at both (t_1,x_1,y_1) and (t_2,x_2,y_2), it follows that $\lambda_j(t_1,x_1,y_1)=\lambda_j(t_2,x_2,y_2)$, $j=1,2,\cdots,m$, and consequently, the functions $\lambda_j(t,x,y)$ can be lifted to the set (t,x,z), $|z|>0$, which proves our assertion.

To conclude, we note that there will be at most two complex λ_j if the order of the equation does not exceed 3. Consequently, there is uniqueness of solutions of the Cauchy problem for second and third order equations with non-multiple characteristics, regardless of the number of independent variables.

Our results on a single equation can be summarized in the following

THEOREM 6. *Let $A(u)=0$ be a linear partial differential equation of order m in a function u of k variables. Let the coefficients of derivatives of order m be real and in C_β, $\beta>1$, and the remaining ones measurable and bounded. Then if the characteristics of the equation are non-multiple,*[5a] *solutions of the Cauchy problem are unique in C_m, provided that $k\neq 3$ or $m\leqq 3$.*

[5a] Although not strictly necessary, here as well as in Theorem 7 we assume the characteristic form to be free of multiplicities in the complex domain.

4. Uniqueness in the case of a system of linear equations. With a few changes, the method used in the preceding section can be applied to a system of equations.

Assume that we have a system of r equations in r functions $u_1, u_2, \cdots, u_r$. Then we regard the functions u_j as the components of a vector function u and the system of equations as a single equation in u with $r \times r$ matrices as coefficients. After bringing the equation to the form (32), we can proceed as before. The v_j in (33) will now be r-vectors and the $v(t)$ in (35) an mr-vector. The matrix $\mathcal{A}(t)$ in (35) will become an $m \times m$ matrix with $r \times r$ matrices as entries, that is, an ordinary $mr \times mr$ matrix. The characteristic polynomial of $-\sigma(\mathcal{A})$ will coincide on $|z| = 1$ with the characteristic form of the system, and the characteristic value functions can be constructed as before.

So far, the way in which we have to modify the argument of the preceding section is obvious. But in trying to construct the matrix N_1 with the property that

$$\sigma(\mathcal{A})N_1 = N_1 D, \tag{43}$$

where D is diagonal, we run into topological difficulties which we know how to overcome only by imposing an additional restriction on the number of independent variables, or else, by assuming that both the system and the u_j are sufficiently differentiable, which enables us to obtain by differentiation another system where such difficulties do not appear. Observe that if (43) is to be satisfied, the column vectors of N_1 must be eigenvectors of $\sigma(\mathcal{A})$ corresponding to its distinct characteristic values. In the case of a single equation, given the simple form of $\sigma(\mathcal{A})$, these eigenvector functions, as we have seen, can be constructed easily in terms of the corresponding characteristic value functions. In the case of a system, though, such an obvious construction does not seem to be available and we are forced to resort to topological techniques. Given a characteristic value function $\lambda_j(t, x, z)$ of $-\sigma(\mathcal{A})$ (in the general case it will be a complex valued function), to each point (t, x, z), $|z| > 0$, there corresponds a one dimensional subspace of the mr-dimensional complex Euclidean space, namely, the corresponding eigenspace of the matrix $\sigma(\mathcal{A})$. Our problem is then to select at each point (t, x, z), $|z| > 0$, a non-zero vector from this eigenspace so as to obtain a continuous eigenvector function corresponding to the characteristic value function $\lambda_j(t, x, z)$. Since the entries of $\sigma(\mathcal{A})$ as functions of (t, x, z) are homogeneous of degree zero with respect to z, it will be sufficient to carry out the construction for $|z| = 1$. Further, since the oscillation of the coeffi-

cients of the equations can be assumed to be arbitrarily small it will be enough to construct the eigenvector function, denote it momentarily by $w_j(t,x,z)$, for $(t,x,z)=(0,0,z)$, $|z|=1$, and then extend it to all (t,x,z), $|z|=1$, by imposing the condition that the Hermitian inner product of $w_j(t,x,z)$ and $w_j(0,0,z)$ be real and positive.

Consider the rm-dimensional complex Euclidean space, its unit sphere S, and the collection P_{rm-1} of all its one dimensional subspaces. Then P_{rm-1} can be regarded as the $(rm-1)$-dimensional complex projective space, and S can be thought of as a fibre bundle over P_{rm-1}. The fibre over each point of P_{rm-1} is the intersection of the corresponding subspace of our Euclidean space with S, i.e., a 1-sphere; the group is the complexes of absolute value 1 (see [12], page 106). We are given a continuous mapping ϕ of the set of points $(0,0,z)$, $|z|=1$, that is, of S_{k-1}, into P_{rm-1} and we want to select continuously a vector in S belonging for each $(0,0,z)$ to the one-dimensional space corresponding to $\phi(0,0,z)$. For this purpose, consider the bundle over S_{k-1} induced by the mapping $\phi: S_{k-1}\to P_{rm-1}$ (see [12], page 47). This induced bundle is a 1-sphere bundle over S_{k-1} with the complexes of absolute value 1 as group. But then, if $k-1\neq 2$, there exists a cross section (see [12], page 99); that is, there exists a continuous mapping $\psi(0,0,z)$ of S_{k-1} into S such that for each z the unit vector $\psi(0,0,z)$ is contained in the one dimensional space corresponding to $\phi(0,0,z)$. This mapping can be assumed, in addition, to be in C^∞ (see [12], page 25).

Once the eigenvector functions of the matrix $\sigma(\mathfrak{A})$ corresponding to each characteristic value function $\lambda_j(t,x,z)$ have been constructed, we can obtain without difficulty the matrix N_1 in (43), and from here on the argument proceeds exactly as in the preceding section.

If the equations in the system have sufficiently differentiable coefficients and the functions $u_1,u_2,\cdots,u_r$ are also sufficiently differentiable it is possible to avoid the topological difficulty discussed above, and consequently, drop the additional restriction $k\neq 3$.

We note that if the functions $u_1,u_2,\cdots,u_r$ and their derivatives of orders less than or equal to $m-1$ vanish on a non-characteristic manifold M and can be further differentiated continuously n times, and if the coefficients of the equations have continuous derivatives up to order $n-1$, then all the derivatives of the u_j of orders less than or equal to $m+n-1$ vanish on M. This means that if we replace our system of equations by another obtained from this one by differentiation, the Cauchy data of the new system will still vanish.

Consider a linear differential operator $A(u)$ of order m with $r\times r$

matrices as coefficients and thought of as acting on vector valued functions $u=(u_1,u_2,\cdots,u_r)$ in the variables $x_1,x_2,\cdots,x_k$. With the notation used in Theorem 5, let a_α be the coefficient of D_α, and let $\pi(A)$ be the $r\times r$ matrix $\pi(A)=\sum_\alpha a_\alpha z^\alpha$, where z^α runs over all monomials in $z_1,z_2,\cdots,z_k$ of degree m. This matrix we shall call the characteristic matrix of A. The polynomial in z with functions of $x=(x_1,x_2,\cdots,x_k)$ as coefficients given by $p(A)=\det[\sum_\alpha a_\alpha z^\alpha]$ coincides with the characteristic form of A.

Let now A_1 and A_2 be two operators as above. Then $\pi(A_1+A_2)=\pi(A_1)+\pi(A_2)\neq 0$ unless A_1+A_2 is of order less than m, in which case, $\pi(A_1)+\pi(A_2)=0$. Further, if A_1A_2 is defined, $\pi(A_1A_2)=\pi(A_1)\pi(A_2)$.

If $\det[\lambda I-\pi(A)]=\lambda^r+b_1\lambda^{r-1}+\cdots+b_{r-1}\lambda+b_r$ is the characteristic polynomial of the matrix $\pi(A)$, then the b_j are homogeneous polynomials in z of degrees jm with functions of x as coefficients. Since a matrix satisfies its own characteristic equation, we have

$$\pi(A)^r+b_1\pi(A)^{r-1}+\cdots+b_{r-1}\pi(A)+b_rI=0. \tag{44}$$

Let now B_j be the differential operator of homogeneous order mj with characteristic matrix b_jI (I being the unit matrix). Then (44) implies that

$$A^r+\sum_1^r B_jA^{r-j}=R$$

is of order less than mr. Consequently, if $A(u)=0$, then

$$B_r(u)=R(u)-(A^{r-1}+\sum_1^{r-1}B_jA^{r-j-1})A(u)=R(u). \tag{45}$$

The matrix B_r is diagonal and its entries on the main diagonal are all identical and equal to an ordinary (in contrast with the matrix differential operators) partial differential operator, denote it by B, which is of homogeneous order rm and whose characteristic form is $\det[\pi(A)]$, that is, coincides with the characteristic form of A. In terms of components, equation (45) can be written as $B(u_j)=R_j(u_1,\cdots,u_r)$, $j=1,2,\cdots,r$, where B is of order mr and has non-multiple characteristics and the R_j are of lower order.

Now the reader will have little difficulty in verifying that the argument used in the preceding section applies also to systems in which the principal part of each equation contains just one function u_j, as is the case with the system above.

We omit further details and summarize our results in the following

THEOREM 7. *Let $A(u)$ be a system of r linear partial differential equations of order m in r functions u_j of k variables. Then if the characteristics of the system are non-multiple:*[5a] i) *solutions of the Cauchy problem are unique in C_m, provided that the coefficients of derivatives of order m are real and in C_β, $\beta > 1$, the remaining ones are measurable and bounded, and $k=2$ or $k>4$;* ii) *solutions of the Cauchy problem are unique in C_{rm}, provided that the coefficients of derivatives of order m are real and in $C_{m(r-1)}$, the remaining ones are in $C_{m(r-1)}$, and $mr \leqq 3$ or $k \neq 3$.*

5. Uniqueness in the non-linear case. The argument by which one extends the results on uniqueness to the non-linear case is well known. We shall briefly discuss the case of a single equation. Systems are treated in exactly the same way.

Let $F(x, u, u_\alpha) = 0$ be a partial differential equation, x denoting the point $x_1, x_2, \cdots, x_k$, and u_α running over the derivatives of the function u of order less than or equal to m, and assume that F is of class C_3.

Let u and v be two solutions with the property that $u - v$ and its derivatives of order less than or equal to $m-1$ vanish on a non-characteristic manifold M, and assume that u and v are of class C_{m+2}. Denote by F_u and F_α the partial derivatives of F with respect to u and u_α respectively. Then $F(x, u, u_\alpha) = F(x, v, v_\alpha) = 0$ and by the mean value theorem,

$$F(x,u,u_\alpha) - F(x,v,v_\alpha) = (u-v)\int_0^1 F_u[x, v+(u-v)s, v_\alpha + (u_\alpha - v_\alpha)s]\,ds$$

$$+ \sum_\alpha (u_\alpha - v_\alpha)\int_0^1 F_\alpha[x, x+(u-v)s, v_\alpha + (u_\alpha - v_\alpha)s]\,ds = 0.$$

If we now replace u, v, u_α and v_α by their values as functions of x in the integrals on the right, we find that $u - v$ satisfies a linear differential equation for which M is non-characteristic. Further, given the differentiability properties of F, u, and v, one verifies readily that the coefficients of this equation are in C_2. From our previous results and assuming non-multiplicity of the characteristics of the equation, it follows that $u - v = 0$ in a neighborhood of M.

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REFERENCES.

[1] N. Aronszajn, "Sur l'unicité du prolongement des solutions des équations aux dérivées partielles elliptiques du second ordre," *Comptes Rendus*, vol. 242 (1956), pp. 723-725.

[2] T. Carleman, "Sur un problem d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes," *Arkiv för Mathematik*, vol. 26B, no. 17 (1939), pp. 1-9.

[3] E. de Giorgi, "Un esempio di non-unicitá della soluzione del problema di Cauchy, relativo ad una equazione differenziale lineare a derivate parziali di tipo parabolico," *Rendiconti di Matematica e della une Applicazioni*, ser. 5, vol. 14 (1955), pp. 382-387.

[4] P. Hartman and A. Wintner, "On the local behaviour of solutions of non-parabolic partial differential equations. III. Approximations by spherical harmonics," *American Journal of Mathematics*, vol. 77 (1955), pp. 453-483.

[5] E. Heinz, "Ueber die Eindeutigkeit beim Cauchyschen Anfangswertproblem einer elliptischen Differentialgleichung zweiter Ordnung," *Nachrichten der Akademie der Wissenschaften in Göttingen*, no. 1 (1955), pp. 1-12.

[6] E. Holmgren, "Ueber Systeme von linearen partiellen Differentialgleichungen," *Ofversigt af Kongl. Vetenskaps Akamemiens Förhandlingar*, vol. 58 (1901), pp. 91-103.

[7] P. D. Lax, "A stability theorem for solutions of abstract differential equations, and its application to the study of the local behaviour of solutions of elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 8 (1956), pp. ————.

[8] C. Müller, "On the behaviour of the solutions of the differential equations $u' = F(x, u)$ in the neighborhood of a point," *Communications on Pure and Applied Mathematics*, vol. 7 (1954), pp. 505-515.

[9] L. Nirenberg, "Uniqueness for the Cauchy problem for differential equations with constant leading coefficients," to appear in the *Communications on Pure and Applied Mathematics*.

[10] A. Plis, "The problem of uniqueness for the solution of a system of partial differential equations," *Bulletin de l'Académie Polonaise des Sciences*, vol. 2 (1954), pp. 55-57.

[11] A. P. Calderón and A. Zygmund, "Singular integral operators and differential equations," *American Journal of Mathematics*, vol. 79 (1957), pp. 901-921.

[12] N. Steenrod, *The topology of fibre bundles*, Princeton, 1951.

$N \times N$ MATRICES OVER AN AW^*-ALGEBRA.†

By S. K. Berberian.

Introduction. The object of this paper is to prove the following conjecture of I. Kaplansky: if A is any AW^*-algebra, the algebra A_n of $n \times n$ matrices over A is AW^*. This result is readily accessible in all but one (namely finite type II) of the types according to which AW^*-algebras are ordinarily classified ([4], Theorems 4.2, 4.6). These known results may be summarized as follows.

After decomposing A into the direct sum of a type I algebra and a continuous algebra (i.e., one which contains no abelian projections), these two cases may be considered separately. If A is type I, then A can be realized as the algebra of all bounded module operators on a suitable AW^*-module H ([6], Theorem 8). Evidently, A_n is the algebra of all bounded module operators on the direct sum of n copies of H and, hence, is an AW^*-algebra of type I ([6], Theorem 7).

Suppose next that A is continuous. After decomposing A as the direct sum of a properly infinite algebra (i.e., one without finite direct summand), and a finite algebra (necessarily of type II), these two cases may be considered separately. If A is properly infinite, one can find n orthogonal projections $e_1, \cdots, e_n$ in A, with sum 1, each equivalent to 1 ([4], Lemma 4.5). By standard matrix unit techniques (see [5], Section 5), A is *-isomorphic with the $n \times n$ matrix algebra $(e_1 A e_1)_n$. Say $w^*w = e_1$, $ww^* = 1$. Then $a \to w^*aw$ is a *-isomorphism of A onto $e_1 A e_1$; this lifts to a *-isomorphism of A_n onto $(e_1 A e_1)_n$, thus A_n is *-isomorphic with the AW^*-algebra A itself.

Thus, we are concerned primarily with the finite type II case. The first step is to show, using the regular ring of A, that A_n is a B_p^*-algebra. At this point Kaplansky showed that A_n is AW^* provided A is a factor; in the general case, our proofs of generalized comparibility and finiteness are modeled on Kaplansky's arguments.

1. Preliminaries. All C^*-algebras to be considered have unity elements. It is appropriate to mention that if B is any C^*-algebra, B_n is also a C^*-

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algebra. This is easily seen by representing B as a uniformly closed *-algebra of operators on a Hilbert space H; then B_n is a uniformly closed *-algebra of operators on the direct sum of n copies of H. This representation shows that if $x = (a_{ij}) \in B_n$, then $\| a_{ij} \| \leqq \| x \|$ for all i, j. On the other hand, $\| x \| \leqq n^2 \max \| a_{ij} \|$, as one sees by expresssing x as the sum of n^2 matrices, each containing at most one non-zero entry. These remarks yield the following useful fact: if (B_α) is any family of C^*-algebras with C^*-sum B, then B_n is the C^*-sum of the $(B_\alpha)_n$ (see [4], Section 2, for a discussion of C^*-sums).

For the remainder of the paper, A is an AW^*-algebra with center Z, n is a fixed natural number, and A_n is the C^*-algebra of $n \times n$ matrices over A. At times we shall identify A with the subalgebra of A_n consisting of the "scalar" matrices $(a\delta_{ij})$, $a \in A$; thus if $x = (a_{ij}) \in A_n$, $ax = (aa_{ij})$ and $xa = (a_{ij}a)$. The center of A_n consists of the scalar matrices $(c\delta_{ij})$, $c \in Z$; thus, when A is identified with the scalar subalgebra of A_n, Z becomes the center of A_n. In particular, the center of A_n is an AW^*-algebra. The first part of the next lemma relates the LUB of central projections to the rest of A_n.

LEMMA 1.1. *Let (h_α) be a family of central projections in the AW^*-algebra A.*

(1) *If $x \in A_n$, $h_\alpha x = 0$ for all α, and $h = \mathrm{LUB}\, h_\alpha$ (computed in Z), then $hx = 0$.*

(2) *If the h_α are orthogonal, and $\mathrm{LUB}\, h_\alpha = 1$, then A_n is the C^*-sum of the $h_\alpha A_n = (h_\alpha A)_n$; if moreover each $h_\alpha A_n$ is AW^*, so is A_n.*

Proof. (1) Say $x = (a_{ij})$. Then $0 = h_\alpha x = (h_\alpha a_{ij})$, thus $h_\alpha a_{ij} = 0$ for all indices; hence $ha_{ij} = 0$. (2) Since A is the C^*-sum of the $h_\alpha A$ ([4], Lemma 2.5), A_n is the C^*-sum of the $(h_\alpha A)_n$ by the earlier remarks. The last assertion of the lemma follows from the fact that the C^*-sum of AW^*-algebras is AW^* ([4], Section 2).

This lemma, of course, justifies the consideration of cases in the Introduction. Another programming for the consideration of cases is as follows. Decompose A into a finite algebra and a properly infinite algebra. After disposing of the properly infinite part by the argument in the Introduction, we may assume A to be finite, and decompose it into parts of type I and type II. The type I case reduces, by Lemma 1.1, to the homogeneous case ([5], Lemma 18), i.e., to the case that A is an $m \times m$ matrix algebra over its center Z ([3], Lemme 4.2); then A_n is the $mn \times mn$ matrix algebra over Z. Thus the finite type I case is reduced to the case that A is commutative, and this is settled as a very special case of ([6], Theorem 7).

2. The regular ring. Henceforth, the AW^*-algebra A is assumed to be finite. Identifying A as a *-subalgebra of its regular ring C ([1], Theorem 2.1), A_n becomes a *-subalgebra of C_n.

LEMMA 2.1. *If $x, y, z \in C_n$, and $x^*x + y^*y + z^*z = 1$, then $x, y, z \in A_n$.*

Proof. Say $x = (x_{ij})$, $y = (y_{ij})$, $z = (z_{ij})$. Then

$$\sum_k (x_{ki}^* x_{ki} + y_{ki}^* y_{ki} + z_{ki}^* z_{ki}) = 1$$

for $i = 1, \cdots, n$; hence $x_{ki}, y_{ki}, z_{ki} \in A$ by ([1], Lemma 5.1).

In particular, $x^*x = 0$ implies $x = 0$, thus C_n is **-regular,* as observed in the proof of ([1], Corollary 7.7). Also, A_n contains all the partial isometries of C_n ([1], Section 11), and A_n is a *B_p*-algebra* ([1], Corollary 7.7). Next, we establish polar decomposition in A_n.

LEMMA 2.2. *Every $x \in A_n$ has a factorization $x = wr$, $r \geqq 0$, $r^2 = x^*x$, $w^*w = RP(x)$, $ww^* = LP(x)$. In particular, $LP(x) \sim RP(x)$.*

Proof. Let r be the positive square root of x^*x in the C^*-algebra A_n. Let $e = RP(x)$, $f = LP(x)$. Since $RP(x) = RP(x^*x) = RP(r^2) = RP(r)$, we have $C_n e = C_n x = C_n r$; hence $x = wr$ for some $w \in C_n$. Since $x = wr = w(er) = (we)r$, we may assume $we = w$. Then $w^*w = e$, $ww^* = f$ by the same argument as in ([2], Lemma 3.3). Quoting ([2], Corollary 3.4), we have

COROLLARY 2.3. *For any two projections $e, f \in A_n$,*

$$e \cup f - f \sim e - e \cap f.$$

COROLLARY 2.4. *For any two projections $e, f \in A_n$, there exist orthogonal decompositions $e = e' + e''$, $f = f' + f''$, such that $e' \sim f'$ and $e''f = f''e = 0$.*

Proof. Set $e' = LP(ef)$, $f' = RP(ef)$, $e'' = e - e'$, $f'' = f - f'$.

Remarks.

(1) Any *sequence* of projections $e_1, e_2, e_3, \cdots \in A_n$ has a LUB. Specifically, LUB $e_i = RP(x)$, where $x = \sum 2^{-i} e_i$ (convergent in norm; see the proof of [7], Theorem 5.8).

(2) To show that A_n is AW^*, it suffices to prove that every family of orthogonal projections (e_i) has a LUB.

3. Generalized comparability. Projections e, f in a C^*-algebra B are said to be *very orthogonal* in case there is a central projection h such that $e \leqq 1-h$, $f \leqq h$. B is said to have *generalized comparability* if, for every pair of projections e, f, there exists a central projection h such that $he \prec hf$ and $(1-h)f \prec (1-h)e$; equivalently, there exist orthogonal decompositions $e = e' + e''$, $f = f' + f''$, such that $e' \sim f'$ and e'', f'' are very orthogonal.

As in the preceding section, A is a finite AW^*-algebra. It will be shown that A_n has generalized comparability. We denote the diagonal matrix units of A_n by $u_1, \cdots, u_n$; thus $u_k = (\delta_{ik}\delta_{kj})$, and $u_1 + \cdots + u_n = 1$. The u_k are, of course, mutually equivalent, and the mapping $a \to (a\delta_{ik}\delta_{kj})$ is a *-isomorphism of A onto $u_k A_n u_k$.

LEMMA 3.1. *Given any projection $e \in A_n$, there exists an orthogonal decomposition $e = e_1 + \cdots + e_n$ such that $e_k \prec u_1$ $(k = 1, \cdots, n)$.*

Proof. We give the proof for $n = 3$. Applying Corollary 2.4 to the pair e, u_1, we have decompositions $e = e_1 + f$, $u_1 = u + v$ with $e_1 \sim u$ and $fu_1 = 0$. Thus $e_1 \prec u_1$ and $f \leqq 1 - u_1 = u_2 + u_3$. Then apply Corollary 2.4 to the pair f, u_2 to obtain decompositions $f = e_2 + e_3$, $u_2 = a + b$, with $e_2 \sim a$ and $e_3 u_2 = 0$. Thus, $e_2 \prec u_2$ and $e_3 \leqq 1 - u_2 = u_1 + u_3$. Since also $e_3 \leqq f \leqq u_2 + u_3$, we have $e_3 \leqq (u_1 + u_3)(u_2 + u_3) = u_3$. Thus $e = e_1 + e_2 + e_3$, where $e_1 \prec u_1$, $e_2 \prec u_2$, $e_3 \leqq u_3$.

LEMMA 3.2. *If $e, f \in A_n$ are projections such that $e \prec u_1$ and $f \prec u_1$, then e, f are generalized comparable.*

Proof. The algebra $u_1 A_n u_1$, being isomorphic with A, has generalized comparability; its projections are simply those of A_n which are $\leqq u_1$. Since the center of A_n is isomorphic with that of $u_1 A_n u_1$ (via the mapping $z \to zu_1$), it is clear that two subprojections of u_1 are generalized comparable in $u_1 A_n u_1$ if and only if they are generalized comparable in A_n.

By assumption, $e \sim e' \leqq u_1$, $f \sim f' \leqq u_1$. If h is a central projection such that $he' \prec hf'$ and $(1-h)f' \prec (1-h)e'$ ([4], Theorem 5.6), evidently $he \sim he' \prec hf' \sim hf$ and $(1-h)f \sim (1-h)f' \prec (1-h)e' \sim (1-h)e$; thus, $he \prec hf$ and $(1-h)f \prec (1-h)e$.

LEMMA 3.3. *A_n has generalized comparability.*

Proof. Given two projections $e, f \in A_n$, we seek decompositions $e = e' + e''$, $f = f' + f''$, such that $e' \sim f'$, and e'', f'' are very orthogonal; thus we are always free to throw away equivalent subprojections of e, f.

We illustrate the proof with $n = 3$. By Lemma 3.1, we have decom-

positions $e = e_1 + e_2 + e_3$, $f = f_1 + f_2 + f_3$, with all summands $\prec u_1$. Applying Lemma 3.2 to the pair e_1, f_1, and throwing away equivalent subprojections, we can assume that e_1, f_1 are orthogonal. Continuing, we may assume every pair e_i, f_j to be very orthogonal; but then it is clear that e, f are very orthogonal.

4. Finiteness. A C^*-algebra is *finite* if the relation $e \sim 1$ implies $e = 1$ ([3], Définition 4.1). If B is a C^*-algebra, a projection $e \in B$ is said to be finite in case the C^*-algebra eBe is finite; equivalently, the relations $f \leqq e$, $f \sim e$ imply $f = e$. We shall show in this section that A_n is finite (we are assuming A is finite).

Using Lemma 1.1(1), the proof of ([4], Theorem 4.2) goes through for A_n, and we conclude that A_n is, at any rate, the direct sum of a finite algebra and an algebra without finite central projections.

LEMMA 4.1. *If A is a finite AW^*-algebra, then A_n is finite.*

Proof. Let $u_1, \cdots, u_n \in A_n$ be as in the preceding section. Since $u_k A_n u_k$ is *-isomorphic with the finite algebra A, u_k is a finite projection. Since $u_1 + \cdots + u_n = 1$, it will suffice to prove the following: if $e, f \in A_n$ are finite projections, so is $e \cup f$. This can be proved exactly as in ([4], Theorem 6.2; the relations in the last line of the proof should read $g \prec e$ and $1 - g \prec 1 - e$). We mention that in the crucial ([4], Lemma 4.5), only countable LUB's are needed.

5. Finite type II. In this section, we assume that the finite AW^*-algebra A is type II.

LEMMA 5.1. *Any projection $e \in A_n$ has an orthogonal decomposition $e = e' + e''$ with $e' \sim e''$.*

Proof. By Lemma 3.1, write $e = e_1 + \cdots + e_n$ with $e_k \prec u_1$ for all k. Say $e_k \sim f_k \leqq u_1$. Since $u_1 A_n u_1$ is a continuous AW^*-algebra, $f_k = f_k' + f_k''$ with $f_k' \sim f_k''$. The equivalence $f_k \sim e_k$ induces a decomposition $e_k = e_k' + e_k''$; the projections $e' = e_1' + \cdots + e_n'$ and $e'' = e_1'' + \cdots + e_n''$ have the required properties.

A non-zero projection e is called *fundamental*, of *order* $k = 2^m$, if there exists a central projection h and an orthogonal decomposition $h = e_1 + \cdots + e_k$ such that $e \sim e_i$ for all i ([3], Définition 6.1). From Lemma 5.1, it results that given any non-zero central projection $h \in A_n$, and any integer $m \geqq 0$, h contains a fundamental projection of order 2^m. By ([3], Lemme

4.4), the order of a fundamental projection, as well as the associated central projection, is unique.

LEMMA 5.2. *Every non-zero projection $e \in A_n$ contains a fundamental projection.*

Proof. Since $e = eu_1 + \cdots + eu_n$, suppose, for example, that $eu_1 \neq 0$. Let $e' = LP(eu_1)$, $f' = RP(eu_1)$, so that $e' \sim f' \leqq u_1$. Since $u_1 A_n u_1$ is isomorphic with A, f' contains a fundamental projection f'' ([3], Lemme 6.4). The equivalence $f' \sim e'$ carries f'' into a fundamental projection e'', $e'' \leqq e' \leqq e$.

LEMMA 5.3. *Every non-zero projection $e \in A_n$ has the form $e = \mathrm{LUB}\, e_i$, with (e_i) a family of orthogonal fundamental projections.*

Proof. Let (e_i) be a maximal family of orthogonal fundamental projections $\leqq e$. If $f \in A_n$ is a projection such that $e_i \leqq f$ for all i, it must be shown that $e \leqq f$. If on the contrary, $x = e(1-f) \neq 0$, then $e(1-f)e = x^*x \neq 0$. By spectral theory, $e(1-f)ey = g$ is a non-zero projection, for suitable $y \in A_n$ ([4], Lemma 2.1). Since $eg = g$, $g \leqq e$. By Lemma 5.2, let h be a fundamental projection, $h \leqq g$. For any i,

$$e_i h = e_i g h = e_i e(1-f)eyh = e_i(1-f)eyh = (e_i - e_i f)eyh = (e_i - e_i)eyh = 0.$$

But $h \leqq g \leqq e$, contradicting maximality.

LEMMA 5.4. *If $(f_j)_{j \in J}$ is a family of orthogonal fundamental projections in A_n, all having the same order, then* LUB f_j *exists.*

Proof. Repeating the arguments in ([3], Lemme 6.10), we obtain a family (h_α) of orthogonal central projections, with $\mathrm{LUB}\, h_\alpha = 1$, such that, for each α, $J_\alpha = \{j \in J : h_\alpha f_j \neq 0\}$ is finite. For any j, there is an α such that $h_\alpha f_j \neq 0$ (Lemma 1.1), thus $J = \bigcup_\alpha J_\alpha$. For each α, set

$$g_\alpha = \sum_{j \in J_\alpha} h_\alpha f_j = h_\alpha \Big(\sum_{j \in J_\alpha} f_j \Big)$$

(these are finite sums). Since A_n is the C^*-sum of the $h_\alpha A_n$ (Lemma 1.1), there exists a (unique) projection $g \in A_n$ such that $h_\alpha g = g_\alpha$ for all α.

We assert that $g = \mathrm{LUB}\, f_j$. First, given an f_j, let us show $f_j \leqq g$, i.e., $f_j - f_j g = 0$. Given an α, it suffices to show that $(f_j - f_j g)h_\alpha = 0$. If $f_j h_\alpha = 0$, this is clear; if $f_j h_\alpha \neq 0$, then $f_j h_\alpha \leqq g_\alpha$; thus

$$(f_j - f_j g)h_\alpha = (f_j h_\alpha) - (f_j h_\alpha)(h_\alpha g) = (f_j h_\alpha) - (f_j h_\alpha) g_\alpha = 0.$$

Next, if f is a projection such that $f_j \leqq f$ for all j, we must show $g \leqq f$

Fixing α, it suffices to show $(g - gf)h_\alpha = 0$, that is, $g_\alpha = g_\alpha f$. For any j, $h_\alpha f_j \leqq h_\alpha f$; hence $g_\alpha \leqq h_\alpha f$, $g_\alpha = g_\alpha h_\alpha f = g_\alpha f$.

LEMMA 5.5. *If $(e_i)_{i \in I}$ is any family of orthogonal projections in A_n, $\mathrm{LUB}\, e_i$ exists.*

Proof. By Lemma 5.3, express e_i as the LUB of orthogonal fundamental projections, say $e_i = \mathrm{LUB}\{e_{ij} : j \in J_i\}$. Let K denote the set of all ordered pairs (i,j), $i \in I$, $j \in J_i$. For $m = 0, 1, 2, \cdots$, let K_m denote the set of all (i,j) for which e_{ij} has order 2^m. Of course, $K = \bigcup_m K_m$. By Lemma 5.4, we may form $f_m = \mathrm{LUB}\{e_{ij} : (i,j) \in K_m\}$. Set $e = \mathrm{LUB}\{f_m : m = 0, 1, 2, \cdots\}$ (countable LUB's are legal). We assert that $e = \mathrm{LUB}\, e_i$.

First, given any i, let us show $e_i \leqq e$. If $j \in J_i$, it will suffice to show $e_{ij} \leqq e$. Indeed, if e_{ij} has order 2^m, $e_{ij} \leqq f_m \leqq e$. Finally, supposing g is a projection such that $e_i \leqq g$ for all i, let us show that $e \leqq g$. For all couples $(i,j) \in K_m$, $e_{ij} \leqq e_i \leqq g$; hence $f_m \leqq g$. Varying m, $e \leqq g$.

THEOREM. *If A is any AW^*-algebra, the algebra A_n of $n \times n$ matrices over A is an AW^*-algebra.*

Proof. According to the discussion in the introduction, we may assume that A is of finite type II. Given any subset $S = \{x_j\}$ of A_n, we must show that the right annihilator of S in A_n is the principal right ideal generated by a projection. The argument for this is given in ([2], Theorem 5.3).

Remarks.

(1) Looking at the scalar subalgebra of A_n, we see that in the passage from A to A_n, type is preserved (although, for homogeneous type I, the order gets multiplied by n).

(2) When A is finite, C_n is, of course, the regular ring of A_n ([1], Section 11).

(3) Let R be any ring with unity, and involution *, and let n be a fixed natural number. Let M be the set of all n-ples $x = (x_1, \cdots, x_n)$, $x_i \in R$, provided with the natural left R-module structure. Then R_n identifies with the ring of all module endomorphisms $T : M \to M$. Assume further that $x_1 x_1^* + \cdots + x_n x_n^* = 0$ implies $x_1 = \cdots = x_n = 0$. Then $[x, y] = x_1 y_1^* + \cdots + x_n y_n^*$ provides M with an R-valued inner product such that $[x, x] = 0$ implies $x = 0$. If T^* is the "conjugate transpose" of the

matrix T, then $[xT, y] = [x, yT^*]$. For a subset N of M, the orthogonal complement $N^\perp$ is defined in the usual way, and $N \cap N^\perp = 0$.

It is straightforward to verify that the following conditions are equivalent:

(a) the right annihilator of any subset of R_n is the principal right ideal generated by a projection;

(b) for any submodule N of M satisfying $N = N^{\perp\perp}$, one has $M = N + N^\perp$.

Replacing R by an AW^*-algebra A, condition (a) is the assertion that A_n is AW^*. Thus, in this case, condition (b) holds, and we have in effect an n-dimensional non-commutative generalization of ([6], Theorems 3, 7).

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REFERENCES.

[1] S. K. Berberian, "The regular ring of a finite AW^*-algebra," *Annals of Mathematics*, vol. 65 (1957), pp. 224-240.

[2] ———, "On the projection geometry of a finite AW^*-algebra," *Transactions of the American Mathematical Society*, vol. 83 (1956), pp. 493-509.

[3] J. Dixmier, "Les anneaux d'opérateurs de classe finie," *Annales Scientifiques de l'École Normale Supérieure*, vol. 66 (1949), pp. 209-261.

[4] I. Kaplansky, "Projections in Banach alegbras," *Annals of Mathematics*, vol. 53 (1951), pp. 235-249.

[5] ———, "Algebras of type I," *Annals of Mathematics*, vol. 56 (1952), pp. 460-472.

[6] ———, "Modules over operator algebras," *American Journal of Mathematics*, vol. 75 (1953), pp. 839-858.

[7] C. E. Rickart, "Banach algebras with an adjoint operation," *Annals of Mathematics*, vol. 47 (1946), pp. 528-550.

POLARIZED VARIETIES, FIELDS OF MODULI AND GENERALIZED KUMMER VARIETIES OF POLARIZED ABELIAN VARIETIES.*[1]

By T. MATSUSAKA.

Introduction. In studying theta-functions and abelian functions, we sometimes fix the set of scalar multiples of a principal matrix attached to the given Riemann Matrix. This implies that we fix one divisor class and scalar multiples of it with respect to homology on the corresponding complex torus. In this paper, we shall introduce the corresponding notion in the abstract case, not only to abelian varieties, but also to arbitrary varieties as done in Weil [22]. In fact, on a complete variety V, non-singular in codimension 1, we shall fix one divisor class, with respect to numerical equivalence, and scalar multiples of it (in general V-divisors shall be said to be numerically equivalent to zero if suitable integral multiples are algebraically equivalent to zero on V). Let us call $\mathfrak{X}$ the set of positive V-divisors contained in them. We shall regard V as carrying a structure we get by putting the set $\mathfrak{X}$ on V (cf. Weil [22], pp. 9-10). We shall call such a variety a *polarized variety*, whenever $\mathfrak{X}$ contains a divisor such that the complete linear system determined by it determines an everywhere biregular birational transformation of V into a projective space. First we shall define the set $\mathfrak{X}$ in a somewhat different way, but we shall see later that it is the same as we mentioned. Projective varieties carry *natural* polarizations, which are determined by the sets containing hyperplane sections. The set of varieties, we get by embedding V into projective spaces by means of complete linear systems determined by divisors in $\mathfrak{X}$, is covered by infinitely many algebraic families of varieties. We shall see a relationship between such algebraic families and algebraic families of divisors contained in $\mathfrak{X}$, which is a content of Section 2.

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We shall follow the same terminology and conventions as in Weil's books (Weil [16], [17]). The writer wishes to express here his thanks to Professor Weil for his valuable advices and criticisms.

In Section 1, we shall state some known results and prove preliminary propositions which we shall need later. In Section 3, we shall prove that the group of everywhere biregular birational transformations of a complete non-singular polarized variety onto itself forms an algebraic group. As a special case, we shall see that the group of automorphisms of a polarized abelian variety is finite. Roughly speaking, the set of everywhere biregular birational transformations of a complete non-singular variety onto itself forms an abstract group, in which one can find the "largest" algebraic group, and to put a structure of polarization on our variety has an effect to cut down the set of cosets, with respect to the algebraic group, to a finite set. In Section 4, we shall prove the existence of the field of moduli of a polarized variety. In the case of elliptic curves, such fields can be generated over the prime field by corresponding j-invariants. In the classical case, when the universal domain is the field of complex numbers, Siegel obtained the result that the field of moduli of a Jacobian variety (polarized by Riemann's theta-divisor in our sense) is the one generated over the complex number field by the corresponding modular functions (cf. Siegel [14]). Finally we shall show the existence of a generalized Kummer variety of a polarized abelian variety over the field of moduli.

Let Z be a positive cycle on an abstract variety U. We shall denote by $|Z|$ the point set attached to Z, i.e. the set of points contained in components of Z. Let U and U' be two varieties with sets of certain structures E and E' respectively. By an *isomorphism* of U with the set of structures E to U' with the set of structures E', we understand an everywhere biregular birational transformation of U to U', mapping E to E'. When $U=U'$, $E=E'$, we say that it is an *automorphism*. We reserve the term "birational transformation" for a general type of transformations with possible fundamental points. When U is a variety defined over a field k, we shall say sometimes that U is *rational* over k.

Let V^n be a complete variety, non-singular in co-dimension 1. We assume that the reader is familiar with linear equivalence and algebraic equivalence of divisors. When two divisors X and X' are linearly equivalent to each other, we write $X \sim X'$. We say that a V-divisor X is *numerically equivalent to* 0 on V if there is a positive integer m such that mX is algebraically equivalent to 0 on V. When V is non-singular, X is usually defined to be numerically equivalent to 0 if and only if $\deg(X \cdot C)=0$ whenever C is a 1-cycle on V such that $X \cdot C$ is defined. When that is so, there is a positive integer m such that mX is algebraically equivalent to 0 on V (cf. Matsusaka [9], Th. 4). Conversely, if mX is algebraically equiv-

alent to 0 on V, we have $\deg(X\cdot C)=0$ whenever C is a 1-cycle on V such that $X\cdot C$ is defined. Therefore the preceding definition of numerical equivalence coincides with the ordinary one when V is non-singular. We denote by $G(V)$, $G_l(V)$, $G_a(V)$ and $G_n(V)$ the group of V-divisors, the group of V-divisors linearly equivalent to 0, the group of V-divisors algebraically equivalent to 0 and the group of V-divisors numerically equivalent to 0 respectively. It is known and easy to see that $G(V)\supset G_n(V)\supset G_a(V)\supset G_l(V)$. We denote by $\mathfrak{L}^*$, $\mathfrak{A}^*$, and $\mathfrak{N}^*$ cosets (divisor classes) of $G(V)$ mod $G_l(V)$, mod $G_a(V)$ and mod $G_n(V)$ and by $\mathfrak{L}$, $\mathfrak{A}$, and $\mathfrak{N}$ the sets of positive divisors contained in $\mathfrak{L}^*$, $\mathfrak{A}^*$ and $\mathfrak{N}^*$. $\mathfrak{L}^*$, $\mathfrak{A}^*$ or $\mathfrak{N}^*$ (resp. $\mathfrak{L}$, $\mathfrak{A}$ or $\mathfrak{N}$) is determined uniquely by one of divisors contained in it. When $\mathfrak{L}^*$, $\mathfrak{A}^*$ or $\mathfrak{N}^*$ (resp. $\mathfrak{L}$, $\mathfrak{A}$ or $\mathfrak{N}$) contains a divisor Z, we denote it sometimes by $\mathfrak{L}^*(Z)$, $\mathfrak{A}^*(Z)$ or $\mathfrak{N}^*(Z)$ (resp. $\mathfrak{L}(Z)$, $\mathfrak{A}(Z)$ or $\mathfrak{N}(Z)$). Let m be a positive integer, then we denote by $m\mathfrak{A}^*$, for instance, the divisor class mod $G_a(V)$ determined by a divisor mY with Y in $\mathfrak{A}^*$. We denote by $m\mathfrak{A}$ the set of positive divisors in $m\mathfrak{A}^*$. We do the same for $\mathfrak{L}^*$, $\mathfrak{L}$ and $\mathfrak{N}^*$, $\mathfrak{N}$.

We can define *specializations of cycles* on abstract varieties. Specializations are compatible with the operations of addition, multiplication and algebraic projection. As far as positive cycles are concerned, specializations are compatible with the operation of intersection-product. Moreover, when the ambient variety is complete and non-singular in co-dimension 1, linear equivalence, algebraic equivalence and numerical equivalence of divisors are preserved by specializations. For the general treatment of specializations of cycles, we refer to the paper by G. Shimura (cf. Shimura [13]). We write $X\to X'$ ref. k, if a cycle X' is a specialization of X over a field k.

Let Z be a positive divisor on a product $W\times V$, k be a common field of definition for W and V over which Z is rational and w be a generic point of W over k. Let $\mathfrak{F}$ be the set of V-divisors which are specializations of $Z(w)$ over k. $\mathfrak{F}$ is usually known as an *algebraic family of positive divisors* on V and (W,Z) is called a *parametrization* of $\mathfrak{F}$ over k. We call $Z(w)$ a generic divisor of $\mathfrak{F}$ over k. When there is a parametrization of $\mathfrak{F}$ over a field k' containing a field of definition of V, we say that $\mathfrak{F}$ is defined over k'. It is known that $\mathfrak{L}$ is an algebraic family of positive divisors (cf. Matsusaka [5], Lemma 4) and is called sometimes a *complete linear system*. The set of rational functions f on V such that $(f)\succ -Z$ forms a module $L(Z)$ and we denote by $l(Z)$ the dimension of the module. $l(Z)-1$ is known as the (geometric) dimension of our complete linear system $\mathfrak{L}(Z)$. In general, let $\mathfrak{M}$ be a finitely generated module of rational functions on V over the universal domain and let $(f_0,\cdots,f_r)$ be a basis of $\mathfrak{M}$. Let k be a common field of

definition for V and for every f_i and let $(u_0, \cdots, u_r)$ be a set of independent variables over k. The set of divisors of the form $(\sum c_i f_i) + (\sum u_i f_i)_\infty$ forms an algebraic family contained in $\mathfrak{L}((\sum u_i f_i)_\infty)$ and it is called a *linear system* or the *linear system determined by* $\mathfrak{M}$. Let x be a generic point of V over k, then we can define a rational mapping of V into an affine space or a projective space by $x \to (f_0(x), \cdots, f_r(x))$. We say that such a mapping is a *rational mapping determined by* $\mathfrak{M}$ or *determined by the corresponding linear system*. We say that it is *non-degenerate*, since $f_0, \cdots, f_r$ are linearly indepenedent. When $\mathfrak{M}$ is the module $\mathfrak{L}(Z)$ itself, we say that it is a rational mapping *determined by* Z. In particular, when the rational mapping is an isomorphism of V into a projective space, we say that it is a non-degenerate *projective embedding* of V. We say that the linear system corresponding to $\mathfrak{M}$ is *ample* if it has no fixed component and determines a non-degenerate projective embedding of V. If a V-divisor X is such that the linear system $\mathfrak{L}(X)$ is ample, we say that X is *linearly effective*. We say that X is *algebraically* (resp. *numerically*) *effective*, when every divisor in $\mathfrak{A}(X)$ (resp. $\mathfrak{N}(X)$) is linearly effective.

Let $\mathfrak{F}$ be an algebraic family of positive V-divisors. We say that $\mathfrak{F}$ is *maximal*, when there is no algebraic family of divisors on V containing $\mathfrak{F}$. We say that $\mathfrak{F}$ is *quasi-complete* when the set $\mathfrak{A}$ containing $\mathfrak{F}$ coincides with $\mathfrak{F}$. We say that $\mathfrak{F}$ is complete if it is quasi-complete and every divisor in it determines a complete linear system of the same dimension. There is an algebraic family $\mathfrak{F}$ of positive V-divisors with the following property: When Z is in $G_a(V)$, there is a divisor X in $\mathfrak{F}$ such that $Z \sim X - X_0$ with a fixed X_0 in $\mathfrak{F}$ (cf. Matsusaka [5], Th. 1). Such a family is called a *total family*. Chow and v. d. Waerden proved, by means of Chow-forms, that $\mathfrak{N}$ is a union of a finite set of maximal algebraic families of positive V-divisors (cf. Chow-v. d. Waerden [4]). Let X be now a positive cycle on a projective space. Then the set of coefficients of the Chow-form of X may be regarded as a set of homogeneous coordinates of a point in a projective space, which we call the *Chow-point* of X. When X is on a product of projective spaces, there is the standard isomorphism of such a product into a projective space, and by means of it we can identify X with a cycle on a projective space. By the Chow-point of such X, we understand the Chow-point of the transformed cycle by the isomorphism.

Finally, let us recall some of the known results on Picard varieties which we need later. Let, as before, V be a complete algebraic variety, non-singular in co-dimension 1 and K be a field of definition for V. Let f be a mapping of $G_a(V)$ into an algebraic variety W with the following

properties: (i) f is well-defined outside of a proper subset of $G_a(V)$; (ii) there is a common field of definition K' for V and W such that whenever Z is in $G_a(V)$ and rational over a field K'' containing K', $f(Z)$ is rational over Z'' if it is defined; (iii) when $Z, Z' \in G_a(V)$, $Z \to Z'$ ref. K' and when f is defined at Z and Z', then $f(Z) \to f(Z')$ ref. K'. If f satisfies the above conditions, we say that f is a *rational mapping* of $G_a(V)$ and that K' is a *field of definition* for f. Moreover, when W is a group variety and f is also a group homomorphism, we say that f is a *rational homomorphism*. There is an abelian variety P and a rational homomorphism f of $G_a(V)$ onto P such that the kernel of f is $G_l(V)$ and that f has the universal mapping property as a rational homomorphism of $G_a(V)$ into an abelian variety. P is called the *Picard variety* of V and f the *canonical rational homomorphism* of $G_a(V)$ onto P. For these, see Chow [3], Matsusaka [5], [6] and Néron-Samuel [11].

Section 1.

1. Let us summarize here some of the known results, which we shall need later in this paper.

THEOREM 1. *Let V be a non-singular variety in a projective space. (i) $G_n(V)/G_a(V)$ is a finite group. (ii) When X is a linearly effective V-divisor and Y is an arbitrary V-divisor, there is a positive integer m_0 with the following properties: Whenever a positive integer m is such that $m > m_0$, $Y + mX$ is numerically effective and $\mathfrak{R}(Y + mX)$ consists of a finite set of complete total families $\mathfrak{A}_1, \cdots, \mathfrak{A}_t$, where t is the order of $G_n(V)/G_a(V)$.*

As to these, see Matsusaka [9], Th. 4 and [7], Th. 2 (cf. also the proof of Matsusaka [7], Lemma 2. There it is proved that $Y + mX$ is linearly effective when m is sufficiently large).

Let A be an abelian variety and X be a divisor on A. We say that X is *non-degenerate* if the set of points a on A such that $X_a \sim X$ is finite. It is easy to see that on every abelian variety there is at least one non-degenerate positive divisor. On a simple abelian variety, every positive divisor is non-degenerate and the existence of such divisors on a general abelian variety can be seen by decomposing it into a product of simple abelian varieties in the sense of isogeny (Weil [17], Th. 28).

THEOREM 2. *Let X be a positive non-degenerate divisor on an abelian variety. There is a positive integer m_0 such that whenever $m > m_0$, mX is linearly effective.* (Cf. Weil [19]).

2.

PROPOSITION 1. *Let A be an abelian variety and $\mathfrak{F}$ be a maximal algebraic family of positive A-divisors. Let X be a divisor in $\mathfrak{F}$, then any divisor X' in $\mathfrak{F}$ is such that $X' \sim X_u$.*

Proof. Let $P(A)$ be the Picard variety of A, f be the canonical homomorphism of $G_a(A)$ onto $P(A)$ and k be a common field of definition for $A, \mathfrak{F}, P(A)$ and for f. Let Y be a generic divisor of $\mathfrak{F}$ over k, then the rational mapping f_Y defined by $f_Y(u) = f(Y_u - Y)$ into $P(A)$ is a homomorphism of A into $P(A)$. Since Y is rational over a regular extension of k, f_Y is defined over a regular extension of k, which implies that f_Y is defined over k by a result of Chow (cf. Chow [2]). Let O be the neutral element of $P(A)$, and let C be the connected component of the algebraic group $|f_Y^{-1}(O)|$ containing the neutral element of A. C is defined over the algebraic closure of k and since $Y_u \sim Y$ whenever u is a point of C, it follows that $X_u \sim X$ for any point u on C ($\sim$ is preserved by specializations!).

There is an abelian subvariety B of A and a homomorphism α from $C \times B$ onto A such that $\alpha(u, v) = u + v$, $B \cap C$ is a finite group and that the kernel of α is a finite group (Weil [17], Prop. 25, Th. 26). We may obviously assume that B, C and α are all defined over k, replacing k by its algebraic closure, if necessary. Put $U = \alpha^{-1}(Y)$, $W = \alpha^{-1}(X)$. Let further $O, d_1, \cdots, d_m$ be all the distinct points in the kernel of α. U and W are obviously invariant by translations T_{d_i}. Let u be a point on A and v be one of the points on $C \times B$ such that $\alpha(v) = u$. Let T be the graph of α. Applying the translation $T_{v,u}$ to $T \cdot (C \times B \times Y)$ on $C \times B \times A$, we get $\alpha^{-1}(Y_u) = U_v$. In the same way, $\alpha^{-1}(X_u) = W_v$. When u is a point on C, denoting by O' the neutral element of B, (u, O') is one of the points on $C \times B$ which are mapped to u by α. Therefore $U_{(u,O')} \sim U$ for any u on C. We claim that U has the projection C on C. Suppose the contrary, then U is of the form $Z \times B$ and $Z_u \sim Z$ for every u on C. This implies that Z is numerically equivalent to zero and since U is positive, it is impossible (cf. Weil [18], p. 123).

Let t and t' be two independent generic points of C over k. We have $U_{(t,O')}(t') \sim U(t')$, and the former is nothing but $U(t' - t)$. Therefore, since t and $t' - t$ are again independent generic points of C over k, we see that $U(t) \sim U(t')$ for any pair of points (t, t') whenever they are defined. This implies that $U \sim C \times M + N \times B$ where M and N are B- and C-divisors respectively. We have $N_t \sim N$ for any t on C and since $U > 0$, we see that $U \sim C \times M$. Let b be a point on B such that $M_b \sim M$. Then $U_{(O'',b)} \sim U$

where O'' is the neutral element of C, and this implies that $sY_b \sim sY$ for a certain positive integer s (Weil [16], Chap. VIII, Th. 7). Hence $Y_{sb} \sim Y$ by Weil [17], Cor. 2, Th. 30. When that is so, sb is contained in $|f_Y^{-1}(O)|$, and since B and $|f_Y^{-1}(O)|$ have only a finite set of points in common, such sb and hence b must form a finite set (cf. Weil [17], Cor. 1, Th. 33). This proves that M is a non-degenerate divisor on B. In the same way, $W \sim C \times M'$, where M' is a B-divisor. Since U and W are algebraically equivalent to each other, so are M and M'. Moreover, since M is non-degenerate, there is a point w on B such that $M \sim M'_w$ and consequently $W_{(O'',w)} \sim U$, because B and its Picard variety have the same dimension (Chow [3], Matsusaka [8]). Going back to A, we see that $sY_w \sim sX$.

Let K be a field containing k over which X is rational and let Y be now a generic divisor of $\mathfrak{F}$ over K. Since $f(Y - X)$ is rational over a regular extension of K, it has a locus E over K on $P(A)$. Let x be a point on A, then since $f(X_x - X)$ is rational $K(x)$, it has a locus E' over K on $P(A)$. The endomorphism $s\delta$ on $P(A)$ is surjective (Weil [17], Cor. 1, Th. 33) and E, E' are mapped onto themselves by the endomorphism. But the relation $sY_w \sim sX$ asserts that the image of E by the endomorphism is contained in E', which proves that $E \subset E'$. Our proposition is thereby proved.

PROPOSITION 2. *Let A be an abelian variety and X be a divisor on A, which we assume to be positive. When X is linearly effective, X is non-degenerate, algebraically effective and $\mathfrak{A}(X)$ forms a complete total family. When $\mathfrak{A}(X)$ contains a total family, X is non-degenerate and $\mathfrak{A}(X)$ is a complete total family.*

Proof. By our definition, when X is linearly effective, we may assume that A is a projective variety and that X is a hyperplane section of A. There is a positive integer m such that $\mathfrak{A}(mX)$ is a complete total family by Th. 1, (ii). If we can show that mX is non-degenerate, then we see immediately that X is non-degenerate by Weil [17], Cor. 2, Th. 30. When that is so, $\mathfrak{A}(X)$ must be a complete total family and every divisor in $\mathfrak{A}(X)$ is linearly equivalent to a translation of X since A and its Picard variety have the same dimension. Moreover when that is so, every divisor in $\mathfrak{A}(X)$ is linearly effective. Therefore, in order to prove our proposition, it is sufficient to prove the following: When $\mathfrak{A}(X)$ contains a maximal total family $\mathfrak{F}$ of divisors, X is non-degenerate.

Let X' be a fixed divisor in $\mathfrak{F}$, then by our Prop. 1, when Y is a divisor in $\mathfrak{F}$, there is a point v on A such that $Y \sim X'_v$. Let $P(A)$ be the Picard variety of A and f be the canonical mapping of $G_a(A)$ onto $P(A)$. Since

$\mathfrak{F}$ is total, any point on $P(A)$ can be written as $f(Y-X')$ with some Y in $\mathfrak{F}$. Therefore, any point on $P(A)$ can be written as $f(X'_v-X')$ with some v on A. Since A and $P(A)$ have the same dimension, it follows now that the set of points u on A with $X'_u \sim X'$ is finite, hence that X' is a non-degenerate divisor. Since $X \sim X'_a$ for some a on A, X is also non-degenerate and our proposition is thereby proved.

Section 2.

3. Let V be a complete variety, non-singular in co-dimension 1, and let X be a positive V-divisor. Let $\mathfrak{X}$ be the set of positive divisors X such that $mX \equiv m'X_0 \bmod G_a(V)$ for a pair (m, m') of positive integers. $\mathfrak{X}$ is uniquely determined when one of the divisors contained in it is given. $\mathfrak{X}$ defines a *structure* on V and when $\mathfrak{X}$ contains a linearly effective divisor, we shall call it a *structure of polarization* (cf. Bourbaki [1], § 8). Any divisor in $\mathfrak{X}$ is called a *polar divisor*. When we regard V as a variety with a structure of polarization, we shall say that V is a *polarized variety* polarized by $\mathfrak{X}$. We shall say that a polarized variety V, polarized by $\mathfrak{X}$, is *defined* over a field k, when V is defined over k and $\mathfrak{X}$ contains a rational divisor over k. For these definitions, see also Weil [22].

When A is an abelian variety, we remark here that it is a variety with an additional structure which we get by putting a point on it (this means that we fix on A one definite law of composition, since the law of composition on A can be determined uniquely up to translations).

Let V be an ordinary variety, E', E'' be two sets of structures and V' (resp. V'') be the variety which we get from V by putting the set of structures E' (resp. E'') on V. When $E' \subset E''$, we say that V' is the *underlying variety* of V'' with the set of structures E'. For instance, the underlying polarized variety of a polarized abelian variety A is the polarized variety which we get from A by removing the structure of the group.

From now on, let us assume that the underlying varieties are non-singular varieties unless the contrary is specifically stated. We can define a *natural* polarization on any projective variety, that is, the polarization defined by the set $\mathfrak{X}$ determined by a hyperplane section of it. Hereafter, let us assume that "a polarized projective variety" means a non-singular projective variety polarized by its natural polarization unless the contrary is specifically stated. Let V be a polarized variety, polarized by a set $\mathfrak{X}$. Let X be a linearly effective divisor in it. The complete linear system $\mathfrak{L}(X)$ defines a non-degenerate projective embedding f_X of V, which is determined uniquely up to projective transformations by $\mathfrak{L}(X)$. Let us assume that V has a set of structures E;

we denote by $f_X(V)$ the transform of V with the set of structures $f_X(E)$. Let $\mathfrak{S}$ be a subset of $\mathfrak{X}$. We shall denote by $\mathfrak{P}(V,\mathfrak{S})$ the set of transforms $f_X(V)$, where the f_X are non-degenerate projective embeddings of V determined by linearly effective divisors X in $\mathfrak{S}$.

4.

LEMMA 1. *Let T be a projective transformation in a projective space defined over a field k and let T^* be a projective transformation in the same space defined by a matrix whose elements are independent variables over k. Then T is a specialization of T^* over k.*

Since the proof is easy, we shall omit it. Such a projective transformation as T^* shall be called a "generic projective transformation" over k.

PROPOSITION 3. *Let W be a variety in a projective space L^n and let $\mathfrak{S}$ be the set of varieties in L which are projectively equivalent to W. Let G be the group variety $PGL(n)$. There is a subvariety X of $G\times W\times L$ such that $(t'\times W\times L)\cdot X$ is defined for every point t' on G, that $\mathrm{pr}_3((t'\times W\times L)\cdot X)$ is in $\mathfrak{S}$ and that every variety in $\mathfrak{S}$ can be obtained in this way.*

Proof. G is a non-singular variety defined over the prime field with its law of composition. Let k be a field of definition for W, t and u be independent generic points of G and L over k. Let T_t be the projective transformation corresponding to t. Let X^* be the locus of $(t,u,T_t(u))$ over k. X^* is a subvariety of $G\times L\times L$ and the projection of X^* on $G\times L$, where L may be any one of the last two factors of $G\times L\times L$, is everywhere regular. Therefore $X^*\cdot(G\times W\times L)$ is defined, is a variety and it is the unique subvariety of X^* lying over $G\times W$ by Weil [16], Chap. IV, Th. 15.

Let t' be a point on G and consider the intersection $(t'\times W\times L)\cap X$ on $G\times W\times L$, where $X=X^*\cdot(G\times W\times L)$. Since the projection of X on $G\times W$ is everywhere regular, the intersection reduces to a variety and it is the uniquely determined subvariety of X having the projection $t'\times W$ on $G\times W$ by the same theorem of Weil. Hence $(t'\times W\times L)\cdot X$ is defined and is of the form $t'\times W_{t'}$, where $W_{t'}$ is the variety having the same dimension as W. We can see easily that $W_{t'}$ is the graph of the restriction of $T_{t'}$ on $W\times T_{t'}(W)$ and so $\mathrm{pr}_3((t'\times W\times L)\cdot X)\in\mathfrak{S}$.

Conversely, let W''' be a variety in $\mathfrak{S}$, then there is a point t'' on G such that $W'''=T_{t''}(W)$. By what we have already observed, $W_{t''}$ is the graph of the restriction of $T_{t''}$ on $W\times T_{t''}(W)$ and therefore

$$W'''=\mathrm{pr}_3((t''\times W\times L)\cdot X).$$

COROLLARY. *Notations being the same as the preceding proposition, let k be a common field of definition for G and W. Let t be a generic point of G over k and x be the Chow-point of $T_t(W)$. x has a locus U over k, which is independent of a choice of a generic point t and also of a choice of a variety W from $\mathfrak{S}$. There is a rational mapping g_W of G onto U when W is fixed in $\mathfrak{S}$ and it has the following properties:*

(i) *g_W is everywhere defined;*

(ii) *$g_W(t')$ is the Chow-point of $T_{t'}(W)$ whenever t' is a point of G;*

(iii) *The set-theoretic image of G by g_W is an open subset of U;*

(iv) *g_W is defined over k.*

Proof. T_t is defined over $k(t)$ which is a regular extension of k and clearly $T_t(W)$ is defined over the same field. Therefore, the Chow-point of $T_t(W)$ is rational over $k(t)$, which implies that x has a locus U over k. Now let W' be another variety in $\mathfrak{S}$. There is a projective transformation T such that $T(W') = W$. Hence $\mathfrak{S}$ is the set of varieties projectively equivalent to W'. Since U is the closure of the set of Chow-points of varieties in $\mathfrak{S}$, U is independent of a choice of W from $\mathfrak{S}$. Since $k(x)$ is a subfield of $k(t)$, there is a rational mapping $g_W = g$ from G onto U such that $g(t) = x$. Let t' be any point on G. By Proposition 3, $\mathrm{pr}_3((t' \times W \times L) \cdot X)$ is defined and consequently x has the uniquely determined specialization over $t \to t'$ ref. k. Therefore g is single-valued at t' and since G is non-singular, it follows that g is defined at t'. Of course $g(t')$ is the Chow-point of $T_{t'}(W)$. Thus, except for (iii), everything is proved.

It is easy to see that U is a pre-homogeneous space on which G is acting, and G acts transitively on the set-theoretic image U' of G by g. Moreover, since it consists of Chow-points of varieties in $\mathfrak{S}$, G cannot transform any point in U' to a point of $U - U'$. This means that U' is the largest subset of U which is biregularly equivalent to the homogeneous space birationally equivalent to the pre-homogeneous space U. Hence U' must be an open set on U (as to pre-homogeneous and homogeneous spaces, see Weil [20]).

5. We can say, by virtue of the corollary to our Proposition 3, that the set of varieties projectively equivalent to a variety in a projective space forms an algebraic family, since we can identify the set with an open algebraic variety. We are going to prove the same for $\mathfrak{P}(V, \mathfrak{A})$, where $\mathfrak{A}$ is a complete algebraic family of divisors on V consisting of algebraically effective divisors. Here we shall prove some preliminary lemmas which we shall need for the purpose.

LEMMA 2. *Let W be a normal variety in a projective space and X, Y be two positive W-divisors. Let k be an algebraically closed field of definition for W and assume that $X \to Y$ ref. k and that $l(X) = l(Y)$. Let g be a non-degenerate rational mapping of W into a projective space determined by $\mathfrak{L}(Y)$, then there is a non-degenerate rational mapping f of W into a projective space such that $(X, f) \to (Y, g)$ ref. k, when $\mathfrak{L}(Y)$ has no base point.*

Proof. Let $r+1$ be the common value of $l(X)$ and $l(Y)$ and U, U' be the Chow-varieties of $\mathfrak{L}(X)$, $\mathfrak{L}(Y)$. Since linear equivalence is preserved by specializations and since U and U' have the same dimension, it follows that $(X, |U|) \to (Y, |U'|)$ ref. k. Replacing k by a bigger field and replacing X by a generic specialization of it over k, if necessary, we may assume that Y, g are rational over k. Then U' is a variety defined over k (Matsusaka [5], Lemma 1). Let $Y_0, \cdots, Y_r$ be divisors in $\mathfrak{L}(Y)$ corresponding to $r+1$ independent generic points $y_0, \cdots, y_r$ of U' over k. There is a set of divisors $(X_0, \cdots, X_r)$ in $\mathfrak{L}(X)$ corresponding to a set of $r+1$ independent generic points $x_0, \cdots, x_r$ of U over a defining field K of U containing k such that

$$(X, |U|, X_0, \cdots, X_r) \to (Y, |U'|, Y_0, \cdots, Y_r) \text{ ref. } k.$$

Since $\mathfrak{L}(Y)$ has no base point, it is easy to see that there is no base point for $\mathfrak{L}(X)$. By our choice of $(X_0, \cdots, X_r)$ and $(Y_0, \cdots, Y_r)$, it follows that sets of functions $(1, h_1, \cdots, h_r)$ and $(1, h'_1, \cdots, h'_r)$ defined by $(h_i) = X_i - X_0$, $(h'_i) = Y_i - Y_0$ form bases of $L(X_0)$ and $L(Y_0)$ respectively. Since $(x) \to (y)$ ref. k, there is a field k' containing k, a generic specialization (x^*) of (x) over k and two sets (z^*), (z') of quantities such that (x^*, z^*) has a locus C of dimension 1 over k' on which (y, z') is a simple rational point over k' (Weil [16], Appendix II, Prop. 7). Replacing everything related to X by a generic specialization of it over k, we may assume that $k' = k$ and we shall omit *. Let us denote the points (x, z) and (y, z') by Q and Q'. Since every X_i is rational over $k(Q)$, we may assume that h_i is defined over $k(Q)$ (Weil [16], Chap. VIII, Cor. 1, Th. 10). Let w be a generic point of W over $k(Q, Q')$. There is a function H_i on $C \times W$ with a field of definition k such that $H_i(Q, w) = h_i(w)$. By multiplying a suitable constant from $k(Q)$ to h_i, we may assume that $Q' \times W$ is not a component of (H_i) for every i. Then $(H_i) \cdot (Q' \times W)$ is defined. We have $(Q \times W) \cdot (H_i)_0 = Q \times X_i$ and $(Q \times W) \cdot (H_i)_\infty = Q \times X_0$ (Weil [16], Chap. VIII, Cor. 1, Th. 4). By the compatibility of specializations with the operation of intersection product, we see that $(Q' \times W) \cdot (H_i)_0 = Q' \times Y_i$ and $(Q' \times W) \cdot (H_i)_\infty = Q' \times Y_0$.

Let f be the rational mapping of W into a projective space defined by $w \to (1, h_1(w), \cdots, h_r(w))$ and g' be the rational mapping of W into the same projective space defined by $w \to (1, h'_1(w), \cdots, h'_r(w))$. Let $T_f, T_g, T_{g'}$ be graphs of f, g, g' and $(Q, T_f) \to (Q', T')$ ref. k be an extension of $Q \to Q'$ ref. k. Since k is algebraically closed, we may take as T' one which is rational over k. H_i is defined at (Q', w), the function h^*_i defined by $H_i(Q', w) = h^*_i(w)$ is defined over k and is such that $(h^*_i) = Y_i - Y_0$. Hence h^*_i and h'_i differ by a constant from k and we may assume that they are actually the same. Let (w', v') be a point on a component of T'. There is a point w^* of W such that $(Q, T_f, w^*, T_f(w^*)) \to (Q', T', w', v')$ ref. k. Assume first that w' is not in $|Y_0|$. Then H_i is defined at (Q', w') and $H_i(Q', w')$ is the uniquely determined specialization of $H_i(Q, w^*)$ over $(Q, w^*) \to (Q', w')$ ref. k. Therefore $(1, H_1(Q', w'), \cdots, H_r(Q', w'))$ must be a representative of v', which is the same as $(1, h'_1(w'), \cdots, h'_r(w'))$. Hence (w', v') must be a point on $T_{g'}$. When w' is on $|Y_0|$, let us choose an index j so that $|Y_j|$ does not contain w' and take H_i/H_j for H_i. Then the same arguments show that (w', v') is on $T_{g'}$. Therefore in any case $|T'|$ must be contained in $|T_{g'}|$ as a point set. Moreover since both have the same dimension and since $T_{g'}$ is a variety, we see that T' must be an integral multiple of $T_{g'}$. By the compatibility of specializations with the operation of algebraic projection, we finally get $T' = T_{g'}$. g and g' differ by a projective transformation F defined over k since k is algebraically closed. Let F^* be a generic projective transformation over a common field of definition for W, C, Q, Q', T_f, T_g and $T_{g'}$ containing k, then $(Q, f, F^*) \to (Q', g', F)$ ref. k. Since F and F^* are isomorphisms, $F \cdot g'$ is the uniquely determined specialization of $F^* \cdot f$ over that specialization with reference to k. Since $F^* \cdot f$ is also a non-degenerate rational mapping of W into the projective space determined by $\mathfrak{L}(X)$, our lemma is proved.

LEMMA 3. *Let U_i $(i = 1, 2, 3)$ be three varieties and X and Y be simple subvarieties of $U_1 \times U_2$ and $U_2 \times U_3$ respectively. Assume that the projection of Y on U_2 is U_2 and is regular at every point of U_2. Then $(X \times U_3) \cdot (U_1 \times Y)$ is defined on $U_1 \times U_2 \times U_3$ and is a cycle of the form $1 \cdot Z$ where Z is a subvariety of $U_1 \times U \times U_3$, which is the only component of $X \times U_3 \cap U_1 \times Y$. When Y is an isomorphism, $\mathrm{pr}_{13}((X \times U_3) \cdot (U_1 \times Y)) = Y \circ X$ is not 0 and $\mathrm{pr}_{13}((X \times U_3) \cdot (U_1 \times Y)) = \mathrm{pr}_{13}((X' \times U_3) \cdot (U_1 \times Y))$ implies $X = X'$ for any cycle X' on $U_1 \times U_2$; and if $\mathrm{pr}_{13}((X \times U_3) \cdot (U_1 \times Y))$ is a birational correspondence (resp. isomorphism), then X is a birational correspondence (resp. isomorphism).*

Proof. Let k be an algebraically closed common field of definition for the U_i, X and Y and for every component of X'. Let f be the rational mapping of U_2 into U_3 with the graph of Y and (x, y) be a generic point of X over k. $(y, f(y))$ is a generic point of Y over k. Let (x', y', z') be any point in $X \times U_3 \cap U_1 \times Y$. (x', y') is then a point on X and (y', z') is a point on Y, and consequently $z' = f(y')$. This shows that the above intersection reduces to the subvariety Z which is the locus of $(x, y, f(y))$ over k. Since x, y and $f(y)$ are simple on U_1, U_2 and U_3 respectively, Z is a simple subvariety of $U_1 \times U_2 \times U_3$. This proves that $(X \times U_3) \cdot (U_1 \times Y) = mZ$ with a positive integer m.

Assume that Y is an isomorphism. Then $k(x, y, f(y))$ is the same as $k(x, f(y))$ and hence the projection Z' of Z on the product $U_1 \times U_3$ has the same dimension as Z and the projection of Z on $U_1 \times U_3$ is regular. Assume that Z' is a birational correspondence. Then $k(x) = k(f(y)) = k(y)$ and this shows that X is a birational correspondence. When Z' is an isomorphism, it is easy to see that X is an isomorphism.

Returning to the general case, we have to prove $m = 1$, and in order to do so, we may assume that every U_i is an affine variety. Let H be a generic hyperplane over k in the ambient affine space of U_1 and (u) be the set of coefficients of the defining equation for H. $H \cdot U_1$ is defined and the intersection-product $(H \cdot U_1 \times U_2 \times U_3) \cdot (X \times U_3)$ is also defined on $U_1 \times U_2 \times U_3$ (Weil [16], Chap. V, Prop. 14), which can be written as $(H \cdot U_1 \times U_2) \cdot X \times U_3$ (cf. Weil [16], Chap. VII, Th. 11) and is prime rational over $k(u)$. Since $(X \times U_3) \cdot (U_1 \times Y)$ and $(X \times U_3) \cdot (H \cdot U_1 \times U_2 \times U_3)$ are defined and since $X \times U_3$, $U_1 \times Y$ and $H \cdot U_1 \times U_2 \times U_3$ intersect properly on $U_1 \times U_2 \times U_3$, we see that

$$((U_1 \times Y) \cdot (X \times U_3)) \cdot (H \cdot U_1 \times U_2 \times U_3) = (U_1 \times Y) \cdot ((H \cdot U_1 \times U_2) \cdot (X \times U_3))$$

by Weil [16], Chap. VII, Th. 10. Moreover, since the projection of $U_1 \times Y$ on $U_1 \times U_2$ is defined everywhere, it follows that

$$(U_1 \times Y) \cdot ((H \cdot U_1 \times U_2) \cdot (X \times U_3))$$

is prime rational over $k(u)$ (Weil [16], Chap. VI, Th. 8), which can also be written as $mZ \cdot (H \cdot U_1 \times U_2 \times U_3)$. Since $Z \cdot (H \cdot U_1 \times U_2 \times U_3)$ is rational over $k(u)$, m must be 1. Hence our assertion about m is proved. The rest of our assertions follow immediately.

6.

LEMMA 4. *Let U be a complete normal variety, and $\mathfrak{A}$ be a complete algebraic family of U-divisors consisting of algebraically effective positive divisors. Let K be a common field of definition for U and $\mathfrak{A}$. Let V be the set of Chow-points of non-degenerate projective embeddings of U determined by divisors in $\mathfrak{A}$, then V is an open algebraic variety in a projective space defined over a field K.*

Proof. Let $\bar{K}$ be the algebraic closure of K and X be a generic divisor of $\mathfrak{A}$ over K. By Lemma 2, the set of Chow-points of non-degenerate projective embeddings determined by divisors in $\mathfrak{A}$, together with their specializations over $\bar{K}$, forms an algebraic variety V' defined over $\bar{K}$. Let v be a generic point of V' over $\bar{K}$, then it is the Chow-point of a non-degenerate projective embedding of U determined by a generic divisor of $\mathfrak{A}$ over K, which we may identify with X. Let f_X be the projective embedding, and T be the graph of it. There is a closed set $\mathfrak{H}$ over $\bar{K}$ on V' such that a specialization T' of T over $\bar{K}$ is irreducible if and only if the Chow-point of T' is in $V'-\mathfrak{H}$; denoting by L^* the ambient space of $f_X(U)$, and denoting by L a hyperplane in L^*, there is a closed subset $\mathfrak{H}'$ over $\bar{K}$ on V' such that T' is not contained in any $U\times L$ if and only if the Chow-point of T' is in $V'-\mathfrak{H}'$ (cf. Chow-v. d. Waerden [4]; v. d. Waerden [15]).

Let again T' be a specialization of T over $\bar{K}$, and now assume that its Chow-point is in $V'-\mathfrak{H}-\mathfrak{H}'=V$. Since T' is irreducible and $\mathrm{pr}_U T'=U$ by the compatibility of specializations with the operation of algebraic projection, T' is the graph of a rational mapping f' of U into L^*. We claim that f' is a non-degenerate projective embedding of U determined by a divisor in $\mathfrak{A}$. Let L be an arbitrary hyperplane in L^* and assume that the Chow-point of T is a generic point of V over a field K' containing $\bar{K}$ over which L is defined. Then we have $((U\times L)\cdot T, T)\to((U\times L)\cdot T', T')$ ref. K', and since we have $X\sim \mathrm{pr}_U((U\times L)\cdot T)$, it follows that $\mathrm{pr}_U((U\times L)\cdot T')\in\mathfrak{A}$. Let x be a generic point of U over K', (y) be a representative of $f'(x)$ and put $y_i=g_i(x)$ $(i=0,1,\cdots,N=\dim L^*)$. Since $(U\times L)\cdot T'$ is defined for all L, it follows that $g_0,\cdots,g_N$ are linearly independent. Since $\mathfrak{A}$ is complete, we have $l(X)=l(\mathrm{pr}_U((U\times L)\cdot T'))$ and this shows that f' is a non-degenerate projective embedding determined by a divisor in $\mathfrak{A}$. Now conversely, if T' is the graph of a non-degenerate projective embedding of U determined by a divisor in $\mathfrak{A}$, the Chow-point of it must be on V' by Lemma 2 and it must not be contained in $\mathfrak{H}$ and in $\mathfrak{H}'$. Our lemma will be proved if we show that V is defined over K.

Since X is a generic divisor of $\mathfrak{A}$ over K, X is rational over a regular extension K^* of K. Since X is rational over K^*, we can find a set of functions $h_0, \cdots, h_N$, all defined over K^*, which forms a basis of $L(X)$ (cf. Weil [16], Th. 10). Hence there is a non-degenerate projective embedding f determined by X defined over K^*. Since any other such embedding can be written as $F \cdot f$, where F is a projective transformation of L^*, it follows that there is a non-degenerate projective embedding f^* determined by X such that it is defined over a regular extension of K and that any other non-degenerate projective embedding determined by a divisor in $\mathfrak{A}$ is a specialization of f^* over $\bar{K}$ by our Lemmas 1, 2. This proves that V' is defined over K. Then it immediately follows that $\mathfrak{H}$ and $\mathfrak{H}'$ are closed subsets of V' over K. Our lemma is thereby proved.

LEMMA 5. *Let V be a variety in a projective space L, h be a rational mapping of V onto a variety W and H be the graph of h. Let $\mathfrak{D}$ be a closed subset of V such that h is defined everywhere on $V - \mathfrak{D}$, and W_0 be the set-theoretic image of $V - \mathfrak{D}$ by h. Let k be a common field of definition for V and h over which $\mathfrak{D}$ is normally algebraic. Then W_0 is a k-open set on W if the following conditions are satisfied:*

(i) *Whenever w' is a point of W_0, $V \times w' \cap H$ contains a component of dimension $t = \dim V - \dim W$, containing a point $v' \times w'$ such that $v' \in V - \mathfrak{D}$;*

(ii) *Denoting by $\mathfrak{F} \times w'$ the maximal algebraic set of dimension t in $V \times w' \cap H$, there is a specialization $(w, h^{-1}(w)) \to (w', Y')$ ref. k such that $v' \in |Y'| \subset \mathfrak{F}$.*

Proof. By Weil [20], Appendix, Prop. 10, W_0 contains a k-open set and the union W' of all such sets is also k-open. Let W^* be the derived normal variety of W with reference to $\bar{k}$ and f be the birational transformation of W to W^* defined by the normalization. Put $h^* = f \cdot h$, then it is a rational mapping of V onto W^* with a field of definition $\bar{k}$. Let x, x' be Chow-points of $h^{-1}(w)$, Y' and M be the locus of x over $\bar{k}$. Since $\bar{k}(w) \supset \bar{k}(x)$, there is a rational mapping g from W onto M such that $g(w) = x$. Put $g^* = g \cdot f^{-1}$. Our assumptions (i), (ii) imply that $w' \times x'$ is an isolated component of $G \cap w' \times M$, where G is the graph of g. Denote by G^* the graph of g^*. Since f^{-1} is defined everywhere on W^* and since any point on W has only a finite set of images by f (cf. Zariski [23]), it follows that there is a point w'^* on W^* with $f^{-1}(w'^*) = w'$ such that $G^* \cap w'^* \times M$ has an isolated com-

ponent $w'^* \times x'$. Hence by the main theorem on birational transformations (cf. Zariski [23]), g^* is defined at w'^*.

Let X be a component of $W - W'$ containing w', assuming that w' is in W_0 but not in W'. There is a subvariety X^* defined over $\bar{k}$ on W^*, going through w'^* such that X is the image of X^* by f^{-1}. Then g^* is defined at a generic point w''^* of X^* over $\bar{k}$ (Weil [16], Chap. IV, Cor., Th. 15). $g^*(w''^*)$ is the Chow-point of a cycle Y'' in L such that $Y'' \to Y'$ ref. $\bar{k}$; $|Y''|$ contains a point v'' such that $(Y'', v'') \to (Y', v')$ ref. $\bar{k}$. Since $v' \notin \mathfrak{D}$, v'' is not contained in $\mathfrak{D}$. Let H^* be the graph of h^*. There is a $\bar{k}$-closed subset $\mathfrak{E}^*_1$ on X^* such that $\mathfrak{E}^*_1$ is the set of points on X^* where g^* is not defined; there is also a $\bar{k}$-closed set $\mathfrak{E}^*_2$ on X^* such that $\mathfrak{E}^*_2$ is the set of points on X^* which have the property that $|H^* \cap V \times w^*_2| \subset |\mathfrak{D} \times W^*|$ (cf. Weil [20], Appendix, Prop. 8; Weil [18], Lemma 7). Since $w''^* \notin \mathfrak{E}^*_1 \cup \mathfrak{E}^*_2 = \mathfrak{E}^*$, $\mathfrak{E}^*$ is a proper $\bar{k}$-closed subset of X^*. Let $\mathfrak{E}$ be the image of $\mathfrak{E}^*$ by f^{-1}, then it is a proper $\bar{k}$-closed subset of X. Let $\bar{w}$ be a point of $X - \mathfrak{E}$. There is a point $\bar{w}^*$ in $X^* - \mathfrak{E}^*$ such that $f^{-1}(\bar{w}^*) = \bar{w}$. Hence g^* is defined at $\bar{w}^*$ and the cycle $\bar{Y}$ in L, whose Chow-point is $g^*(\bar{w}^*)$, contains a point $\bar{v}$ in $V - \mathfrak{D}$. Since $|\bar{Y} \times \bar{w}^*| \subset |H^* \cap V \times \bar{w}^*|$, it follows that $|\bar{Y} \times \bar{w}| \subset |H \cap V \times \bar{w}|$. Hence h is defined at $\bar{v}$ and $h(\bar{v}) = \bar{w}$. This proves that $W' \subsetneqq W' + X - \mathfrak{E} \subset W_0$. Let α be any automorphism of $\bar{k}$ over k. Since $X - \mathfrak{E} \subset W_0$, it follows that $(X - \mathfrak{E})^\alpha \subset W_0$. Consequently there is a k-open set W'' on W such that $W' \subsetneqq W'' \subset W_0$, which contradicts to our choice of W' and our lemma is thereby proved.

7. Let U be a complete variety, non-singular in co-dimension 1, and let $\mathfrak{X}$ be a set which can put a structure of polarization on U. Let f_X be a non-degenerate projective embedding of U determined by a linearly effective divisor X in $\mathfrak{X}$, and denote by $c(f_X(U))$ the Chow-point of $f_X(U)$, except when U is an abelian variety. When U is an abelian variety with the neutral element O, denoting by U' the underlying variety of the abelian variety U, we understand by $c(f_X(U))$ a pair $(x, f_X(O))$ of points, where x is the Chow-point of $f_X(U')$. Let $\mathfrak{S}$ be a subset of $\mathfrak{X}$, then we shall say that $\mathfrak{P}(U, \mathfrak{S})$ has a *structure of an algebraic variety* when the set of points $c(f_X(U))$, with $f_X(U)$ in $\mathfrak{P}(U, \mathfrak{S})$, forms an open algebraic variety. When that is so, we shall say that $\mathfrak{P}(U, \mathfrak{S})$ is defined over a field k if the open algebraic variety is defined over k.

THEOREM 3. *Let U be a non-singular variety in a projective space and $\mathfrak{A}$ be a complete family of algebraically effective positive U-divisors. Then*

$\mathfrak{P}(U,\mathfrak{A})$ *has a structure of an algebraic variety. The same is true when* U *is an abelian variety. In both cases,* $\mathfrak{P}(U,\mathfrak{A})$ *is defined over a common field of definition for* U *and* $\mathfrak{A}$; *moreover* $\mathfrak{P}(U,\mathfrak{A})$ *has the smallest field of definition if* $\mathfrak{A}$ *is total.*

Proof. Let us assume first that U is an abelian variety and let U' be the underlying variety of U. We shall show that $\mathfrak{P}(U,\mathfrak{A})$ has a structure of an algebraic variety when $\mathfrak{P}(U',\mathfrak{A})$ has a structure of an algebraic variety. Let W be the set of points $c(f(U'))$, where f is a non-degenerate projective embedding of U' determined by a divisor in $\mathfrak{A}$. Assume that W is an open algebraic variety and let k be a field of definition for W. When w is a generic point of it over k, it is the Chow-point of a variety of the form $f(U')$ in $\mathfrak{P}(U',\mathfrak{A})$, which is defined over $k(w)$. By Weil [16], Chap. VII, Th. 12, there is the uniquely determined subvariety Z on the product $W\times L$ of W with the ambient space of $f(U')$ such that $Z(w)=f(U')$. Let w' be a point on W and let $f'(U')$ be the corresponding variety in $\mathfrak{P}(U',\mathfrak{A})$. Let u' be a point on $f'(U')$, then there is a non-degenerate projective embedding g' of U' determined by a divisor in $\mathfrak{A}$ such that $g'(U')=f'(U')$ and that when 0 is the neutral element of U, $g'(O)=u'$. Hence $g'(U)$ is the abelian variety with the neutral element u' whose underlying variety is $g'(U')$, which proves that $c(g'(U))=(w',u')$. Conversely, let f'' be a non-degenerate projective embedding of U determined by a divisor in $\mathfrak{A}$. $f''(U)$ is an abelian variety with the neutral element $f''(O)$ and let w'' be the Chow-point of the underlying variety of $f''(U)$. Then by our assumption, w'' is a point of W and hence $(w'',f''(O))$ is a point of Z. Thus $|Z|$ coincides with the set of points $c(f''(U))$. Our assertion is thereby proved.

Let us return to our general case when U is an ordinary non-singular variety in a projective space. Let k be a common field of definition for U and $\mathfrak{A}$, then the set of Chow-points of non-degenerate projective embeddings determined by divisors in $\mathfrak{A}$ forms an open algebraic variety $V-\mathfrak{D}$ in a projective space defined over k (cf. Lemma 4), where V is a projective variety, and $\mathfrak{D}$ is a k-closed set on V. Let v be a point on $V-\mathfrak{D}$ and f be the corresponding projective embedding of U. When w is the Chow-point of $f(U)$, we shall write U_w, f_w, t_w for $f(U), f, v$, for the sake of simplicity. Let us now assume that v is a generic point of V over k. Since $k(v)\supset k(w)$, w has a locus W over k, which is the closure of the point set $\{c(U^*)\}$ with U^* in $\mathfrak{P}(U,\mathfrak{A})$, and there is a rational mapping h of V onto W such that $h(v)=w$. Let U^* be a variety in $\mathfrak{P}(U,\mathfrak{A})$, a be the Chow-point of U^* and t_a be one of the points on $V-\mathfrak{D}$ such that the corresponding non-degenerate

embedding f_a transforms U to $U^* = U_a$. Let t_w, t_s be two independent generic points of V over $k(t_a, a)$ and let k' be the algebraic closure of $k(t_a, t_s)$. Let f_w, f_s be corresponding non-degenerate projective embeddings of U to t_w, t_s; put $f_{w,s} = f_w \cdot f_s^{-1}$, $f_{s,w} = f_s \cdot f_w^{-1}$ and let $t_{w,s}, t_{s,w}$ be Chow-points of $f_{w,s}, f_{s,w}$ respectively. Let t'_w be a generic point of $h^{-1}(w)$ over $k'(w)$, f'_w be the corresponding embedding of U and let t'_s be the Chow-points of $f_{s,w} \cdot f'_w$. By Lemma 3, t'_s is also on $V - \mathfrak{D}$ and let Z be the locus of $(t_w, w, t_{s,w}, t'_w, t'_s)$ over k'. Z is then a subvariety of $V \times W \times L \times V \times V$ where L is the ambient space of $t_{s,w}$.

Let $\mathfrak{F}$ be the projection of $(t_a, a) \times L \times V \times V \cap Z$ on the fourth factor of the product. We claim that $\mathfrak{F}$ has the following three properties: (i) Every component of $\mathfrak{F}$ has the same dimension as $h^{-1}(w)$; (ii) Every component of $\mathfrak{F}$ is not contained in $\mathfrak{D}$; (iii) $\mathfrak{F}$ is the closure on V of the set of points t'_a on V corresponding to non-degenerate embeddings of U, transforming U to U_a. Let (t_a, a, u, t, x) be a point in $(t_a, a) \times L \times V \times V \cap Z$. By Lemma 3 and by the compatibility of specializations with the operations of intersection-product and algebraic projection, we see that u is the Chow-point of $f_s \cdot f_a^{-1} = f_{s,a}$ since $t_{s,w}$ is the Chow-point of $f_s \cdot f_w^{-1}$. Let $T_{s,a}$ be the graph of $f_{s,a}$ and T be the cycle whose Chow-point is t. Since t'_s is the Chow-point of $f_{s,w} \cdot f'_w$, it follows that x is the Chow-point of $T_{s,a} \circ T$. Since t'_s is contained in $|h^{-1}(s)|$, it follows that x is also contained in it. By Lemma 3, the correspondence $T \to T_{s,a} \cdot T$ gives a one-to-one correspondence between points of $\mathfrak{F}$ and points of a subset of $|h^{-1}(s)|$. This shows that the dimension of each component of $\mathfrak{F}$ is at most that of $h^{-1}(s)$. Since f_s, f_a are defined over k', $f_{s,a} = f_s \cdot f_a^{-1}$ is defined over k' and consequently the Chow-point $u = t_{s,a}$ of $f_{s,a}$ is rational over k'. Therefore we get $\dim_{k'}(u, t, x) = \dim_{k'}(t)$ by Lemma 3. Since $Z \cap (t_a, a) \times L \times V \times V$ has not any component of lower dimension than that of $Z \cap (t_w, w) \times L \times V \times V$ by Weil [16], Chap. IV, Prop. 26, which is obviously the same as $\dim h^{-1}(w) = \dim h^{-1}(s)$, our statement (i) is proved. By Lemma 3, if $T_{s,a} \circ T$ is an isomorphism, so is T. Therefore, every component of $\mathfrak{F}$ contains the Chow-point of an isomorphism of U onto U_a and since $\mathfrak{A}$ is complete, it must be non-degenerate. This proves (ii). Let t'_a be a point on $V - \mathfrak{D}$ such that the corresponding embedding f'_a transforms U to U_a. In order to prove (iii), it is sufficient to prove that the projection of $Z \cap (t_a, a) \times L \times V \times V$ on the last factor of the product is $|h^{-1}(s)|$, since the Chow-point of $f_{s,a} \cdot f'_a$ is on $|h^{-1}(s)|$ and since the correspondence $f_s \to f_{s,a}^{-1} \cdot f_s$ determines a one-to-one correspondence between point of $|h^{-1}(s)|$ and points of a subset of $\mathfrak{F}$ as before for the fixed $f_{s,a}$. By Lemma 3, the projection of $\bar{Z} \cap (t_w, w) \times L \times V \times V$ on the fifth factor

is $|h^{-1}(s)|$, hence the same is true for $Z \cap (t_a, a) \times L \times V \times V$. (iii) is thereby proved.

Let V' be the locus of (t_w, w) over k', then since $h(t_w) = w$, V' is also the locus of (t_w, w) over k. Let $\mathfrak{D}'$ be the set of points (t', c) on V' such that $t' \in \mathfrak{D}$. Then $\mathfrak{D}'$ is also a closed subset of V' over k. Let h' be the rational mapping of V' onto W defined by $h'(t_w, w) = w$. h' is defined everywhere on $V' - \mathfrak{D}'$ and the set-theoretic image W_0 of $V' - \mathfrak{D}'$ by h' is the point-set we are interested in. We are going to show that to V', $\mathfrak{D}'$ and h', our Lemma 5 can be applied. Let $(t_w, w, h'^{-1}(w)) \to (t_a, a, Y)$ ref. k', where we assume now that a is a point on W_0. We have $h'^{-1}(w) = h^{-1}(w) \times w$, and since $\mathrm{pr}_4(Z \cdot ((t_w, w) \times L \times V \times V)) = h^{-1}(w)$, $|Y|$ is contained in $\mathfrak{F} \times a$. Since $\dim Y = \dim h^{-1}(w) = \dim \mathfrak{F}$ by (i), every component of Y contains a point of $V' - \mathfrak{D}'$ by (ii). Moreover, we see from this that every component of $Y \times a$ is a component of $H' \cap V' \times a$, where H' is the graph of h'. Therefore we can apply our Lemma 5 and thus W_0 is an open algebraic variety over k in a projective space. Thus the first half of our theorem is proved.

Let k_0 be the smallest field of definition for W. We shall show that W_0 is defined over k_0, which will settle the last half of our theorem. Let w be a generic point of W over k_0 and f_w be a non-degenerate embedding of U into the projective space, transforming U to U_w. It is easy to see that $\mathfrak{P}(U, \mathfrak{A}) = \mathfrak{P}(U_w, f_w(\mathfrak{A}))$. $f_w(\mathfrak{A})$ contains a rational divisor over $k_0(w)$, a hyperplane section of U_w, and since $f_w(\mathfrak{A})$ is complete and total, it is easy to see that $f_w(\mathfrak{A})$ is defined over a purely inseparable extension of $k_0(w)$. Let us take for $U, \mathfrak{A}$ the variety U_w and the family $f_w(\mathfrak{A})$. Since W and W_0 are invariants of $\mathfrak{P}(U, \mathfrak{A}) = \mathfrak{P}(U_w, f_w(\mathfrak{A}))$, it follows that $W - W_0$ is normally algebraic over $k_0(w)$. Let w' be a generic point of W over $k_0(w)$. Then in the same way, we can show that $W - W_0$ is normally algebraic over $k_0(w')$, hence $W - W_0$ is normally algebraic over k_0. Our theorem is thereby proved completely.

Section 3.

8.

PROPOSITION 4. *Let U be a variety in a projective space, non-singular in co-dimension 1, and $\mathfrak{X}$ be a set of divisors on U which defines a polarization on it. There is a divisor class $\mathfrak{N}^*$ with respect to numerical equivalence with the following property: A positive U-divisor X belongs to $\mathfrak{X}$ if and only if X belongs to $m\mathfrak{N}^*$ for a certain positive integer m.*

Proof. By Néron's result, $G(U)/G_a(U)$ is finitely generated (Néron [10]). On the other hand, $G_n(U)/G_a(U)$ is a subgroup of $G(U)/G_a(U)$, consisting of elements of finite orders. Therefore, there is a finite set $(D_1, \cdots, D_s)$ of divisors on U, which are independent modulo $G_n(U)$, such that any U-divisor X can be written as $X \equiv \sum c_i D_i \bmod G_n(U)$. Let Y be a divisor in $\mathfrak{X}$, then expressing Y in the form $Y \equiv \sum a_i D_i \bmod G_n(U)$, let b be the greatest common divisor of $a_1, \cdots, a_s$. Putting $a_i = b \cdot a'_i$, let $\mathfrak{N}^*$ be the coset of $G(U) \bmod G_n(U)$ determined by $Z = \sum a'_i D_i$. Let m be a positive integer and X be a positive divisor in $m\mathfrak{N}^*$. Clearly X is contained in $\mathfrak{X}$ by the definition of $\mathfrak{X}$. Conversely, let X be any divisor in $\mathfrak{X}$. There are two positive integers m and t such that $mY \equiv tX \bmod G_n(U)$ and that m and t are relatively prime to each other. Then $bmZ \equiv tX \bmod G_n(U)$. Expressing X as $X \equiv \sum c_i D_i \bmod G_n(U)$, we see that $\sum (bma'_i - tc_i) D_i \equiv 0 \bmod G_n(U)$, which implies that $bma'_i \equiv 0 \bmod t$. Therefore $mb \equiv 0 \bmod t$ since $(a'_1, \cdots, a'_s)$ is a set of relatively prime integers. Our proposition is thereby proved.

THEOREM 4. *Let V be a complete non-singular variety, polarized by a set $\mathfrak{X}$. $\mathfrak{X}$ can be written as $\bigcup \mathfrak{A}(X)$ with X in $\mathfrak{X}$. Let $\mathfrak{X}^*$ be the set of divisors X in $\mathfrak{X}$ such that $\mathfrak{A}(X)$ is complete, total and consists of algebraically effective divisors. Then $\mathfrak{X}^*$ can also be written as $\bigcup \mathfrak{A}(X)$ with X in $\mathfrak{X}^*$. Every $\mathfrak{A}(X)$ in the expression for $\mathfrak{X}$, except for a finite set of them, is also a component of the same expression for $\mathfrak{X}^*$. $\mathfrak{P}(V, \mathfrak{X})$ (resp. $\mathfrak{P}(V, \mathfrak{X}^*)$) is the union of $\mathfrak{P}(V, \mathfrak{A}(X))$, where $\mathfrak{A}(X)$ is a component of $\mathfrak{X}$ (resp. $\mathfrak{X}^*$). Moreover, when V is an abelian variety, $\mathfrak{P}(V, \mathfrak{X})$ and $\mathfrak{P}(V, \mathfrak{X}^*)$ coincide.*

Proof. The first two statements about expressions for $\mathfrak{X}$ and $\mathfrak{X}^*$ are immediate consequences of definitions.

Since $\mathfrak{X}$ contains a linearly effective divisor by our definition, we may assume that $\mathfrak{X}$ is determined by a hyperplane section of a variety V in a projective space. Let C be a hyperplane section of V. There is the divisor class $\mathfrak{N}^*$ with respect to numerical equivalence having the property described in Proposition 4. Let Z be a divisor in $\mathfrak{N}^*$, then there is a positive integer s such that $C \equiv sZ \bmod G_n(V)$. By Theorem 1, (ii), there is a positive integer m_0 such that whenever $m > m_0$, $\mathfrak{N}(iZ + mC)$ is covered by a finite set of complete total families and that $iZ + mC$ is numerically effective for $0 \leqq i \leqq s-1$. Put $t = m_0 s$, then $m_0 C \equiv tZ \bmod G_n(V)$. We shall show that $\mathfrak{N}(t'Z)$ is not empty, consists of numerically effective divisors and is covered by a finite set of complete total families whenever $t' > t$. In fact,

$t'Z \equiv atZ + bsZ + cZ \bmod G_n(V)$, where a is a positive integer, b and c are non-negative integers and c is less than s. Therefore we have $t'Z \equiv (am_0 + b)C + cZ \bmod G_n(V)$ which should be numerically effective and $\mathfrak{N}(t'Z)$ is covered by a finite set of complete total families by our choice of m_0. Therefore our third assertion is proved.

Our fourth assertion is trivial. In order to prove the last assertion, it is sufficient to show that when X is a linearly effective divisor on an abelian variety, $\mathfrak{A}(X)$ is complete, total and X is algebraically effective, which are consequences of Proposition 2.

THEOREM 5. *Let V be a complete non-singular variety, polarized by a set $\mathfrak{X}$. Then every polarized variety in $\mathfrak{P}(V, \mathfrak{X})$, polarized by the natural polarization, is isomorphic to each other. When V is an abelian variety, underlying varieties of polarized abelian varieties in $\mathfrak{P}(V, \mathfrak{A})$ are projectively equivalent to each other, where $\mathfrak{A}$ is a divisor class of $V \bmod G_a(V)$ contained in $\mathfrak{X}^*$.*

Proof. The first statement is obvious. Assume that V is an abelian variety and A, A' be two polarized abelian varieties in $\mathfrak{P}(V, \mathfrak{A})$, polarized by natural polarizations, with neutral elements O, O'. Let us identify A, A' with their underlying varieties. There is an isomorphism f between two polarized abelian varieties A, A', which is determined by a divisor on A, algebraically equivalent to a hyperplane section C of A. Let Y be such a positive divisor on A, then $f(Y)$ is a hyperplane section of A', and $f(C) \equiv f(Y) \bmod G_a(A')$. Since $f(C)$ is non-degenerate by Proposition 2, there is a point a' on A' such that $f(C)_{a'} \sim f(Y)$. Hence, denoting by T the translation on A' defined by a', we have $T \cdot f(C) \sim f(Y)$. Moreover, $T \cdot f$ is an isomorphism between two polarized varieties A, A' and transforms hyperplane sections to hyperplane sections. Since linear systems of hyperplane sections on A and A' are complete, $T \cdot f$ must be a projective transformation of A onto A'. Our theorem is thereby proved.

9. Let U be a complete non-singular variety, polarized by a set $\mathfrak{X}$. We shall denote by $\mathfrak{G}(U, \mathfrak{X})$ the group of automorphisms of the polarized variety U. Let W be also a complete non-singular polarized variety, polarized by a set $\mathfrak{Y}$ and assume that f is an isomorphism between polarized varieties U and W. Then it is easy to see that the correspondence $g \to f \cdot g \cdot f^{-1}$, $g \in \mathfrak{G}(U, \mathfrak{X})$ gives an isomorphism of $\mathfrak{G}(U, \mathfrak{X})$ and $\mathfrak{G}(W, \mathfrak{Y})$.

PROPOSITION 5. *Let U^n be a projective variety, non-singular in codimension 1, and G^r be a connected algebraic group variety. Let Z^{n+r} be a*

subvariety of $G\times U\times U$ and assume that the projection of Z on $G\times U$ is $G\times U$ and is everywhere regular, where U is any one of the second and the third factors of $G\times U\times U$. Then for any point g on G, $(g\times U\times U)\cdot Z$ is defined and $Z(g)$ is the graph of an automorphism of U. Further, denote by Y^g the transform of a U-cycle Y by $Z(g)$ and assume that $u^e=u$ for any point u on U, where e is the neutral element of G. Then we have the following: (i) *When X is a U-divisor, $X^g\equiv X \bmod G_a(U)$;* (ii) *When X is a U-divisor such that $X\equiv 0 \bmod G_a(U)$, $X^g\sim X$.*

Proof. Let $g\times T$ be a component of $g\times U\times U\cap Z$, which is simple on $G\times U\times U$. Let (u,v) be a point on T; by our definition of Z, v is uniquely determined by g and u, hence $\dim T\leqq \dim U$, which proves our assertion. Then it is clear that $Z(g)$ is the graph of an automorphism of U. Next, we claim that $Z\cdot(G\times X\times U)=Z'$ is defined on $G\times U\times U$. Let (g,x,u) be any point in $Z\cap G\times X\times U$, then $u=x^g$ and this shows that the dimension of any component of our intersection must be at most $r+n-1$, which proves our assertion. Therefore by Weil [16], Chap. VII, Cor., Th. 10, we see that

$$((g\times U\times U)\cdot Z)\cdot(G\times X\times U)$$
$$=(g\times U\times U)\cdot(Z\cdot(G\times X\times U))=g\times Z(g)\cdot(X\times U).$$

$Z\cdot(G\times X\times U)$ can be regarded as a cycle on $G\times L\times U$, where L is the ambient space of U, then by Weil [16], Chap. VII, Th. 18, we see that

$$\{(g\times L\times U)\cdot(Z\cdot(G\times X\times U))\}_{G\times L\times U}$$
$$=(g\times U\times U)\cdot(Z\cdot(G\times X\times U)).$$

That being so, by Weil [16], Chap. VII, Th. 16, we have

$$\mathrm{pr}_{13}((g\times U\times U)\cdot(Z\cdot(G\times X\times U)))=(g\times U)\cdot\mathrm{pr}_{13}(Z\cdot(G\times X\times U)).$$

Therefore

$$(g\times U)\cdot\mathrm{pr}_{13}(Z\cdot(G\times X\times U))=g\times\mathrm{pr}_3(Z(g)\cdot(X\times U))=g\times X^g.$$

By taking g to be e, we get $(e\times U)\cdot\mathrm{pr}_{13}(Z\cdot(G\times X\times U))=e\times X^e=e\times X$, which proves that $X^g\equiv X \bmod G_a(U)$. (i) is thereby proved.

Let $\mathfrak{F}$ be a total maximal family of positive U-divisors, V be the Chow variety of $\mathfrak{F}$ (the set of Chow-points of divisors in $\mathfrak{F}$) and denote by Y_v the divisor whose Chow-point is $v\in V$. Let $P(U)$ be the Picard variety of U and γ be the canonical homomorphism of $G_a(U)$ onto $P(U)$. Let k be an algebraically closed common field of definition for $U, G, Z, V, P(U)$ and γ

over which X is rational. By (i), $Z(g)^{-1}(Y_v) \equiv Y_v \bmod G_a(U)$ and hence the mapping $(g, v) \to \gamma(Z(g)^{-1}(Y_v) - Y_v)$ defines a rational mapping of $G \times V$ into $P(U)$ with a field of definition k (Y_v is rational over $k(v)$; cf. W. L. Chow, "On the defining field of a divisor in an algebraic variety," *Proc. Amer. Math. Soc.*, vol. 6, 1950), which we shall denote by F. Assume that g and v are independent generic points of G and V over k, then F can be written as $F(g, v) = F_1(g) + F_2(v)$, where F_1 is a rational mapping of G into $P(U)$ and F_2 is a rational mapping of V into $P(U)$ (cf. Weil [17], Cor., Th. 7), both defined over k. Since $F(g, v)$ has the uniquely determined specialization $\gamma(Z(e)^{-1}(Y_v) - Y_v)$ over $(g, v) \to (e, v)$ ref. $k(v)$, and since (e, v) is a simple point on $G \times V$, F is defined at (e, v) (cf. Weil [16], Chap. VI, Th. 13). Therefore, we have

$$F(e, v) = \gamma(Z(e)^{-1}(Y_v) - Y_v) = Z(Y_v - Y_v) = O,$$

which proves that $F_2(v) = -F_1(e)$ is a constant, rational over k. By the compatibility of specializations with the operations of intersection-product and algebraic projection, we see that F is single-valued everywhere on $G \times V$ and has the value $\gamma(Z(g)^{-1}(Y_w) - Y_w)$ at (g, w). Hence we have

$$\gamma(Z(g)^{-1}(Y_v) - Y_v) = \gamma(Z(g)^{-1}(Y_w) - Y_w),$$

whenever w is a generic point of V over $k(g, v)$. Consequently

$$\gamma(Z(g)^{-1}(Y_t) - Y_t) = \gamma(Z(g)^{-1}(Y_s) - Y_s)$$

for any two points t, s on V. Let now X be a U-divisor such that $X \equiv 0 \bmod G_a(U)$, then since $\mathfrak{F}$ is total, there are two points t and s on V such that $Y_t - Y_s \sim X$. By what we have proved, we have $(Y_t - Y_s)^g \sim Y_t - Y_s$ and since $X^g \sim (Y_t - Y_s)^g$, it follows that $X^g \sim X$. Our proposition is thereby proved.

10.

LEMMA 6. *Let U be a normal variety in a projective space, C be a hyperplane section of U and assume that $\mathfrak{A}(C)$ is a complete algebraic family of algebraically effective divisors and that the linear system of hyperplane sections on U is complete. Let Σ be the set of automorphisms f of U such that $f(C) \in \mathfrak{A}(C)$, and W be the set of Chow-points of f. Then $W = \bigcup_0^m W_i$, where the W_i are open algebraic varieties in a projective space. When U and $\mathfrak{A}(C)$ are defined over a field k, then W is normally algebraic over k.*

Proof. Let V be the set of Chow-points of graphs of non-degenerate

projective embeddings determined by divisors in $\mathfrak{A}(C)$. Then by our Lemma 4, V is an open algebraic variety in a projective space L. Let f be an automorphism of U such that $f(C) \in \mathfrak{A}(C)$. Then since f is everywhere biregular, $f^{-1}(C) \in \mathfrak{A}(C)$; since $\mathfrak{A}(C)$ is complete, every divisor in it determines a complete linear system of the same dimension as C. Therefore, f is a non-degenerate projective embedding determined by $f^{-1}(C)$ and the Chow-point of it is on V. Let k be a common field of definition for U and $\mathfrak{A}(C)$, then V is an open algebraic variety in L over k by Lemma 4. Let v be a generic point of V over k and T_v be the variety with the Chow-point v. T_v is the graph of a non-degenerate projective embedding of U, determined by a generic divisor of $\mathfrak{A}(C)$ over k and is defined over the field $k(v)$. Therefore, there is a subvariety T of $V \times U \times \bar{L}$, where $\bar{L}$ is the ambient space of U, such that $T(v) = T_v$ and that k is a field of definition for T (cf. Weil [16], Chap. VII, Th. 12). Let $\mathfrak{F} = T \cap V \times U \times U$. $\mathfrak{F}$ is a closed algebraic set, normally algebraic over k on $V \times U \times \bar{L}$. Let w be a point on V and T_w be the corresponding subvariety of $U \times \bar{L}$, then since T_w is the uniquely determined specialization of T_v over $v \to w$ ref. k, it follows that $T \cap w \times U \times \bar{L}$ reduces to a component $w \times T_w$. When T_w is the graph of an automorphism of U, $w \times T_w$ is a subvariety of $V \times U \times U$ and hence $\mathfrak{F} \cap w \times U \times \bar{L}$ contains $w \times T_w$. Conversely, assume that $\mathfrak{F} \cap w \times U \times \bar{L}$ contains a component $w \times R$ of at least the same dimension as U. Then since $w \times R$ is a subvariety of T, we have $w \times R \subset w \times T_w$. Since $\dim R \geqq \dim T_w$, it follows that $R = T_w$ and the projection of T_w on $\bar{L}$ is U. Therefore T_w must be the graph of an automorphism of U. Thus we have shown that the set of points w on V, such that T_w is the graph of an automorphism of U, coincides with the set of points on V such that $w \times U \times \bar{L} \cap \mathfrak{F}$ contains a component of dimension at least equal to that of U. Such a set forms a closed algebraic set on V, normally algebraic over k by Weil [18], Lemma 7. Our lemma is thereby proved.

LEMMA 7. *Maintaining the same notations and assumptions as in the preceding lemma, denote also by T_{w_i} the subvariety of $U \times U$ with the Chow-point w_i on W_i. Let W_0 be a component of W containing the Chow-point of the diagonal of $U \times U$. Then we have the following results:*

(i) $\dim W_0 = \dim W_1 = \cdots = \dim W_m$;

(ii) $W_i \cap W_j = \emptyset$ *whenever* $i \neq j$;

(iii) *Let w_i and w_j be points of W_i and W_j respectively, then $T_{w_j} \circ T_{w_i}$ is the graph of an automorphism of U belonging to* $\mathfrak{S}$;

(iv) *Let k_{ij} be a common field of definition for W_i and W_j containing k, w_i and w_j be independent generic points of W_i and W_j over k_{ij} and let W_s be a component of W containing the Chow-point w_s of $T_{w_j} \circ T_{w_i}$, then s is a function $\alpha(i,j)$ of (i,j) only and (w_i, w_j, w_s) has a locus Z_{ijs} over k_{ij};*

(v) *Projections of Z_{ijs} on $W_i \times W_j$, $W_i \times W_s$ and on $W_j \times W_s$ are regular and surjective;*

(vi) *Let v_i, v_j, v_s be arbitrary points on W_i, W_j, W_s respectively, then $(v_i, v_j) \times W_s \cap Z_{ijs}$, $v_i \times W_j \times v_s \cap Z_{ijs}$ and $W_i \times (v_j, v_s) \cap Z_{ijs}$ have single zero-dimensional components;*

(vii) $\alpha(0,j) = \alpha(j,0) = 0$.

Proof. By Lemma 3, $T_{w_j} \circ T_{w_i}$ is defined and is again the graph of an automorphism of U, since T_{w_i}, T_{w_j} are such. Since $T_{w_i}(C) \equiv T_{w_j}(C) \equiv C \bmod G_a(U)$, we see that $T_{w_j} \circ T_{w_i}(C) \equiv C \bmod G_a(U)$. (iii) is thus proved. Let now w_i and w_j be independent generic points of W_i and W_j over k_{ij}, then the Chow-point w_s of $T_{w_j} \circ T_{w_i}$ is rational over $k_{ij}(w_i, w_j)$ by Lemma 3. Hence (w_i, w_j, w_s) has a locus Z_{ijs} over k_{ij} on $W_i \times W_j \times W_s$. Let (v_i, v_j) be an arbitrary point on $W_i \times W_j$, then $T_{v_j} \circ T_{v_i}$ is the uniquely determined specialization of $T_{w_j} \circ T_{w_i}$ over $(w_i, w_j) \to (v_i, v_j)$ ref. k_{ij} by Lemma 3 and by Lemma 6. This shows that $\alpha(i,j)$ actually depends only on i and j as soon as we prove (ii). Moreover, we have shown that $Z_{ijs} \cap (v_i, v_j) \times W_s$ reduces to a point and the projection of Z_{ijs} on $W_i \times W_j$ is regular and surjective. Next let us show, among other things, that W_0 is the unique component of W containing the Chow-point of Δ. Let W_i be a component of W and put $\alpha(0,i) = j$. Since the projection of Z_{0ij} on W_j contains W_i because $T_{v_i} \circ \Delta = T_{v_i}$, it follows that $W_i = W_j$ proving that $\alpha(0,i) = i$. In the same way, $\alpha(0,i) = i$. Suppose now that W_i contains also the Chow-point of Δ. Then W_i contains W_0 since $\alpha(0,i) = i$ and consequently $W_i = W_0$.

The formula $\alpha(0,i) = i$ shows that $\dim W_0 \leqq \dim W_i$ by virtue of Lemma 3. Conversely, let w_i be a generic point of W_i over k_{0i}. Then the set of Chow-points of $T_{w_i}^{-1} \circ T_{v_i}$, where v_i is a point on W_i, is contained in $|W|$ and forms an open algebraic variety in the ambient space of W. Since it contains the Chow-point of Δ, it must be contained in W_0 by what we have seen, proving that $\dim W_0 \geqq \dim W_i$ (cf. Lemma 3). Thus (i) is proved.

Let a be a common point of W_i and W_j and w_i, w_j be independent generic points of W_i, W_j over $k_{ij}(a)$ respectively. Since Δ is a specialization of both $T_{w_i} \circ T_a^{-1}$, $T_{w_j} \circ T_a^{-1}$ over $k_{ij}(a)$, it follows that Chow-points of them are on W_0. Since $\dim W_0 = \dim W_i = \dim W_j$, it follows that $T_{w_i} \circ T_a^{-1}$. $T_{w_j} \circ T_a^{-1}$

are generic specializations of each other over $k_{ij}(a)$, which shows that T_{w_i} and T_{w_j} are generic specializations of each other over k_{ij}. Therefore $W_i = W_j$ and (ii) is thus proved. Now (v) and (vi) are easy consequences of what we have proved.

11.

THEOREM 6. *Let U be a complete non-singular polarized variety, polarized by a set $\mathfrak{X}$. Then there is an algebraic group $G = \bigcup_0^m G_i$, where G_0 is the connected component of G containing the neutral element e and the G_i are cosets of G_0, and subvarieties Z_i of $G_i \times U \times U$ with the following properties:*

(i) *Projections of Z_i on the product of G_i and the second factor and also on that of G_i and the third factor are everywhere regular;*

(ii) *$(g_i \times U \times U) \cdot Z_i = g_i \times Z_i(g_i)$ is defined for all g_i on G_i and in particular $Z_0(e) = \Delta$;*

(iii) *$Z_i(g_i) = Z_j(g'_j)$ implies $i = j$ and $g_i = g'_j$;*

(iv) *The group of automorphisms $\mathfrak{G}(U, \mathfrak{X})$ of U is the set of $Z_i(g_i)$ for all g_i on G_i and for all i* (we identify rational mappings and graphs);

(v) *When $g_j \cdot g_i = g_s \in G_s$, then $Z_s(g_s) = Z_j(g_j) \circ Z_i(g_i)$;*

(vi) *When our polarized variety U is defined over a field k_0 and when $\mathfrak{X}$ contains a rational divisor X_0 over k_0 which is linearly effective, then G is normally algebraic over k_0 and $\Sigma_i Z_i$ is rational over a finite purely inseparable extension of k_0.*

Proof. Let C be a linearly effective divisor in $\mathfrak{X}$ and k be a field of definition of U over which C is rational. By Weil [16], Chap. VIII, Th. 10, there is a non-degenerate projective embedding of U determined by C, defined over k. By our Th. 1, when r is sufficiently large, $\mathfrak{A}(rC)$ is a complete total algebraic family of algebraically effective divisors, and since it contains a rational divisor rC over k, $\mathfrak{A}(rC)$ is defined over k (cf. Matsusaka, [6], Prop. 1). Let f be any automorphism of U belonging to the set $\mathfrak{G}(U, \mathfrak{X})$, then $f(C) \equiv C \bmod G_n(U)$; since $G_n(U)/G_a(U)$ is a finite group (cf. Th. 1), $f(sC) \equiv sC \bmod G_a(U)$ whenever s is a positive integral multiple of the order of $G_n(U)/G_a(U)$. Therefore, we may assume, from the beginning, that U is a non-singular projective variety defined over k, $\mathfrak{X}$ contains a hyperplane section C of U, $\mathfrak{A}(C)$ is a complete total family of algebraically

effective divisors and that every automorphism f in $\mathfrak{G}(U,\mathfrak{X})$ transforms C to a divisor in $\mathfrak{A}(C)$.

When that is so, f is a non-degenerate projective embedding of U determined by $f^{-1}(C)$ and our Lemmas 6, 7 can be applied. Hence the set W of Chow-points of $f \in \mathfrak{G}(U,\mathfrak{X})$ can be written as $\bigcup_0^m W_i$, where the W_i are open algebraic varieties in a projective space, and has properties described in Lemmas 6, 7. Let k' be the smallest perfect field containing k, then the W_i are defined over separable algebraic extensions k'_i of k'. Let G'_i be a derived normal variety of W_i with reference to k'_i and let G_i be the graph of the inverse of our normalization $W_i \to G'_i$. It is easy to see that the G_i can be so chosen that they are in the same space and that when γ is an automorphism of our universal domain over k, $W_i{}^\gamma = W_j$ implies $G_i{}^\gamma = G_j$. Then $\bigcup_0^m G_i$ forms an algebraic set normally algebraic over k. Since the k'_{ii}/k' are separable algebraic extensions, $\bigcup_0^m k'_i/k'$ is also a separable algebraic extension. Since the G_i are normal with reference to the k'_i, it follows that the G_i are still normal with reference to $K = \bigcup_0^m k'_i$ and the $G_i \times G_j$ are also normal with reference to K (cf. Weil [20], Appendix, Prop. 14). Therefore, if we assume (i), (ii), (iii), (iv), $\bigcup_0^m G_i$ is an algebraic group satisfying (v) by our Lemma 7, such that G_0 is the connected component of it containing the neutral element.

Let w_i be a generic point of W_i over k'_i and g_i be the corresponding point on G_i. Let T_{w_i} be the variety whose Chow-point is w_i, then it is the graph of an automorphism in $\mathfrak{G}(U,\mathfrak{X})$. Then T_{w_i} is defined over the field $k'_i(w_i) = k'_i(g_i)$ and hence there is a subvariety Z_i of $G_i \times U \times U$ defined over k'_i such that $Z_i(g_i) = T_{w_i}$ (cf. Weil [16], Chap. VII, Th. 12). Z_i is then the locus of $(g_i, u, T_{w_i}(u))$ over k'_i, where u is a generic point of U over $k'_i(g_i)$. (ii) is then an immediate consequence of the facts that the birational transformation $g_i \to w_i$ of G_i to W_i is everywhere defined on G_i and that specializations are compatible with the operation of intersection-product. Since the birational transformation $w_i \to g_i$ of W_i to G_i is such that any point on W_i has only a finite set of image points in G_i, (i) follows easily because $G_i \times U \times U$ is non-singular (cf. Weil [16], Chap. VI, Th. 13). In particular, when g'_i is a point on G_i and w'_i is the corresponding point on W_i, we have $Z_i(g'_i) = T_{w_i}$, and conversely, if w'_i is a point on W_i and g'_i is one of the corresponding points on G_i, we have $Z_i(g'_i) = T_{w_i}$. Therefore, the set of $Z_i(g'_i)$ with g'_i on G_i coincides with the set of varieties whose Chow-points are on W_i. Since any f in $\mathfrak{G}(U,\mathfrak{X})$ is a non-degenerate projective embedding of U determined by a divisor $f^{-1}(C)$, and conversely, any automorphism f' of U, such that $f'^{-1}(C)$ is in $\mathfrak{A}(C)$, is in $\mathfrak{G}(U,\mathfrak{X})$, it

follows that $\mathfrak{G}(U,\mathfrak{X})$ coincides with the set of $Z_i(g'_i)$ for all g'_i on G_i and for all i. Hence (iv) is proved. (v) is an immediate consequence of our Lemma 7. Since $G=\bigcup_0^m G_i$ is normally algebraic over k, we see that $\sum Z_i$ remains unchanged by any automorphism of the universal domain over k, which proves (vi).

Let w_i be a point on W_i and assume that g_i, g'_i are two corresponding points on G_i. Let w_0 be a generic point of W_0 over $K(w_i,g_i,g'_i)$ and let g_0 be the corresponding point on G_0. Let Z_{0ii} be the subvariety of $W_0\times W_i\times W_i$ defined in our Lemma 7. Then by Lemam 7, (v) and (vi), $Z_{0ii}\cap(w_0\times w_i\times W_i)$ reduces to a point (w_0,w_i,v_i) and v_i is a generic point of W_i over K. Let h_i be the point on G_i corresponding to v_i. Then we see immediately that $g_i\cdot g_0=g'_i\cdot g_0=h_i$, which proves $g_i=g'_i$. Since $W_i\cap W_j=\emptyset$ whenever $i\neq j$, and since we have taken for G_i the graph of our normalization $W_i\to G'_i$, it follows that $G_i\cap G_j=\emptyset$ whenever $i\neq j$. Our statement (iii) follows now immediately.

COROLLARY 1. *Let A be a polarized abelian variety, polarized by a set $\mathfrak{X}$. Then $\mathfrak{G}(A,\mathfrak{X})$ is a finite group.*

Proof. Let V be a variety and Z be a subvariety of $V\times A\times A$ such that $Z(v)$ defined by $(v\times A\times A)\cdot Z=v\times Z(v)$ is the graph of an automorphism f_v of the underlying polarized variety of A. Let O be the neutral element of A and put $f_v(O)=O_v$. Then $f_v-O_v=\alpha$ is an automorphism of our polarized abelian variety A. Let k be an algebraically closed common field of definition for A, for V and for Z. Then since α is an automorphism of A defined over $k(v)$, we see that α is defined over k, by taking v to be a generic point of V over k (cf. Chow [2]). Therefore, when v' is a point on V such that $Z(v')$ is defined and is the graph of an automorphism of the underlying polarized variety of A, $f_{v'}-O_{v'}=\alpha$. Our corollary now follows immediately from our theorem.

COROLLARY 2. *Maintaining the same notations and assumptions as our theorem, G_0 is a normal subgroup of G. Denoting by $P(U)$ the Picard variety of U and by β the canonical homomorphism of $G_a(U)$ onto $P(U)$, there is a rational homomorphism F of G_0 into $P(U)$ defined by $F(g)=\beta(Z_0(g)^{-1}(X)-X)$ with a fixed U-divisor X. F depends only on the class of X with respect to algebraic equivalence. When we take X to be a linearly effective divisor on U, the kernel H_0 of F is a subgroup of a projective general linear group, and H_0 is a normal subgroup of G_0.*

Proof. Let K be a common field of definition for U, for every G_i, Z_i and for $P(U)$ and β. Since G_0 is a group variety of finite index in G, it follows that G_0 is a normal subgroup of G. Let X be a U-divisor and K' be a field containing K over which X is rational. Let g be a generic point of G_0 over K' and let F be the mapping of G_0 into $P(U)$ defined by

$$F(g) = \beta(Z_0(g)^{-1}(X) - X).$$

Then F is an everywhere defined rational mapping. F is then a rational homomorphism up to a constant (cf. Weil [17], no. 19). Since the neutral element e of G_0 is such that $Z_0(e) = \Delta$, it follows that F itself is a homomorphism. Let Y be another U-divisor such that $X \equiv Y \bmod G_a(U)$, and F' be the similar homomorphism defined in terms of Y. Since we have $Z_0(g')^{-1}(X - Y) \sim (X - Y)$ for any g' on G_0 by our Prop. 5, it follows that $F = F'$. Let us now assume that X is linearly effective. Then since the kernel H_0 of F consists of those g' such that $Z_0(g')^{-1}(X) \sim X$, it follows that H_0 can be identified with an algebraic subgroup of the group of projective transformations $PGL(m)$, where $m = l(X) - 1$. Our corollary is thereby proved.

Let U be a non-singular projective variety and G be an abstract group of automorphisms of U. Then by Proposition 5 and Theorem 6, we see that *G contains the largest connected algebraic subgroup and to put a structure of polarization on U has an effect of cutting down the set of cosets with respect to the subgroup to a finite set.* The author found this when Weil pointed out that the original proof of our Corollary 1 can be generalized to our Theorem 6.

Section 4.

12. We shall define the field of moduli of a polarized variety (resp. abelian variety) V. Let $\mathfrak{P}(V, \mathfrak{X}^*) = \bigcup \mathfrak{P}(V, \mathfrak{A})$, where $\mathfrak{X}^*$ is the subset of $\mathfrak{X}$ defined in our Theorem 4. Then the *field of moduli* of V is the smallest field K with the following properties:

(i) One of the $\mathfrak{P}(V, \mathfrak{A})$ in the above expression for $\mathfrak{P}(V, \mathfrak{X}^*)$ is defined over K;

(ii) Every other $\mathfrak{P}(V, \mathfrak{A})$ in the above expression for $\mathfrak{P}(V, \mathfrak{X}^*)$ is defined over a separably generated extension of K.

We are going to prove some lemmas, which we shall need to prove the existence theorem.

LEMMA 8. *Let V be a complete non-singular variety (resp. abelian variety) and $\mathfrak{A}$, $\mathfrak{A}'$ be complete families of V-divisors consisting of algebraically effective positive divisors. Let K be a field of definition for $\mathfrak{P}(V,\mathfrak{A})$. Then $\mathfrak{P}(V,\mathfrak{A}')$ is defined over an algebraic extension of K.*

Proof. Without loss of generality, we may assume that K is algebraically closed. First let us assume that V is a complete non-singular variety, and X, Y be divisors in $\mathfrak{A}, \mathfrak{A}'$ respectively. Since K is algebraically closed, there is a variety V' contained in $\mathfrak{P}(V,\mathfrak{A})$, defined over K. There is an isomorphism f between V and V' such that $\mathfrak{A}$ is transformed to the complete algebraic family on V' containing a hyperplane section of V'. Also $\mathfrak{A}'$ is transformed to the complete algebraic family on V' containing $f(Y)$ by f. Since $\mathfrak{A}(f(Y))$ is a complete family and since K is algebraically closed, $\mathfrak{A}(f(Y))$ is defined over K. Since $\mathfrak{P}(V,\mathfrak{A}') = \mathfrak{P}(V',\mathfrak{A}(f(Y)))$, our lemma follows from Theorem 3 in this case. When V is an abelian variety, our lemma follows easily from the definitions and from what we have proved.

LEMMA 9. *Let V be a complete non-singular variety (resp. abelian variety) and X be a positive algebraically effective divisor. Assume that $\mathfrak{A}(X)$ and $\mathfrak{A}(mX)$ are complete total families of V-divisors, where m is a positive integer. Let K and K' be the smallest fields of definitions of $\mathfrak{P}(V,\mathfrak{A}(X))$ and $\mathfrak{P}(V,\mathfrak{A}(mX))$ respectively. Then K' is a subfield of K.*

Proof. Since $\mathfrak{P}(V,\mathfrak{A}(X))$ has a structure of an algebraic variety over K by Theorem 3, we can speak of a generic variety of $\mathfrak{P}(V,\mathfrak{A}(X))$ over K. Let V^* be a generic variety of it over K. There is an isomorphism f^* of V onto V^* such that $f^*(\mathfrak{A}(X))$ is a complete total family of V^*-divisors containing hyperplane sections of V^*. Hence $f^*(\mathfrak{A}(X))$ contains a rational divisor over any field of definition of the underlying variety of V^*. Let K^* be a regular extension of K, over which V^* is defined. Then since $f^*(\mathfrak{A}(X))$ contains a rational divisor over K^*, and since it is complete and total, it is defined over K^* (Matsusaka [6], Prop. 1). $f^*(\mathfrak{A}(mX))$ is the complete total family on V^* containing $f^*(mX)$ and hence it also contains a rational divisor over K^*. Therefore $f^*(\mathfrak{A}(mX))$ is also defined over K^*. Since $\mathfrak{P}(V,\mathfrak{A}(mX)) = \mathfrak{P}(V^*,f^*(\mathfrak{A}(mX)))$, it follows that $\mathfrak{P}(V,\mathfrak{A}(mX))$ is defined over K^*. On the other hand, $\mathfrak{P}(V,\mathfrak{A}(mX))$ is defined over an algebraic extension of K by Lemma 8, therefore it is defined over K and our lemma is thereby proved.

LEMMA 10. *Let U be a non-singular variety in a projective space and X be a positive U-divisor such that $\mathfrak{A}(X)$ and $\mathfrak{A}(mX)$ are total quasi-complete*

families consisting of algebraically effective divisors, where m is a positive integer. Let K be a common field of definition for U and $\mathfrak{A}(mX)$. If m is prime to the characteristic p, $\mathfrak{A}(X)$ is defined over a separably algebraic extension of K.

Proof. In order to prove our lemma, we may identify K with its separably algebraic closure. Since every element in the algebraic closure $\bar{K}$ of K is purely inseparable over K, $\mathfrak{A}(X)$ has no other conjugate than itself over K since it is quasi-complete. Let K' be a purely inseparable extension of K over which $\mathfrak{A}(X)$ is defined, and X^* be a generic divisor of $\mathfrak{A}(X)$ over K'. Without loss of generality, we may identify X^* with X. Let $K'(u)$ be the smallest field containing K' over which X is rational. There is a positive integer s such that $K(u^{p^s})$ is a regular extension of K, where u^{p^s} is the point whose coordinates are the p^s-th power of the coordinates of u. Also there is a positive integer t such that p^tX is rational over $K(u^{p^s})$. Since m and p^t are relatively prime, there are two integers a and b such that when Y is a generic divisor of $\mathfrak{A}(mX)$ over $K(u^{p^s})$, $aY + bp^tX \equiv X \bmod G_a(U)$. Moreover, $aY + bp^tX$ is rational over a regular extension of K by our choice of Y. We claim that $\mathfrak{L}^*(aY + bp^tX)$ contains a positive divisor. Since $aY + bp^tX - X$ is algebraically equivalent to 0, there is a positive divisor X' in $\mathfrak{A}(X)$ such that $aY + bp^tX - X \sim X' - X$. Therefore, $\mathfrak{L}^*(aY + bp^tX)$ contains a positive divisor, and we may assume that X' is rational over the same regular extension of K over which $aY + bp^tX$ is rational (cf. Weil [16], Chap. VIII, Cor. 1, Th. 10). Since $\mathfrak{A}(X)$ contains a rational divisor X' over a regular extension of K, and since it is quasi-complete total family, $\mathfrak{A}(X)$ is defined over a regular extension of K (Matsusaka [6], Prop. 1). Therefore $\mathfrak{A}(X)$ must be defined over K and our lemma follows now from Theorem 3.

13.

THEOREM 7. *Let V be a complete non-singular variety (resp. abelian variety), polarized by a set $\mathfrak{X}$ and assume either that the order of $G_n(V)/G_a(V)$ is prime to the characteristic p or isomorphisms between V', V'' in $\mathfrak{P}(V, \mathfrak{X})$ are defined over separably generated extensions of any common field of definition for V', V''. Then there is the field of moduli of V. When the characteristic is 0, it has the additional property that any field of definition of $\mathfrak{P}(V, \mathfrak{A})$, which appears in the expression for $\mathfrak{P}(V, \mathfrak{X}^*)$, contains the field of moduli.*

Proof. Let r be the order of $G_n(V)/G_a(V)$. By Proposition 4, there is the divisor class $\mathfrak{N}^*$ on V with respect to $G_n(V)$ such that X belongs to $\mathfrak{X}$ if and only if X is positive and belongs to $m\mathfrak{N}^*$ for a certain positive integer m. There is a positive integer m_0 such that whenever $m' > m_0$, $m'\mathfrak{N}$ is not empty, contains a numerically effective divisor and is covered exactly by r complete total families (cf. Th. 1). Let us fix an integer m' with that property. Let $m'\mathfrak{N}$ be covered by complete total families $\mathfrak{A}(X_i)$. Let us denote by k_i the smallest field of definition of $\mathfrak{P}(V, \mathfrak{A}(X_i))$. Put $K^* = k_1 \cup \cdots \cup k_r$ and let $k_{i,m}$ be the smallest field of definition for $\mathfrak{P}(V, \mathfrak{A}(mX_i))$, where m is a positive integer. We shall show that K^* contains $k_{i,m}$ and is a finite normal extension of the latter, satisfying $[K^* : k_{i,m}]_s \leqq r$.

By Lemma 9, k_i contains $k_{i,m}$ and the former is an algebraic extension of the latter by Lemma 8. Since there is a positive integer a such that aX_i and aX_j are algebraically equivalent to each other on V, and since $\mathfrak{A}(aX_i)$ is a complete total family consisting of numerically effective divisors by our choice of m', k_j is an algebraic extension of $k_{i,a}$, again by Lemmas 8 and 9. If we take a to be a multiple of m, $k_{i,m}$ itself is an algebraic extension of $k_{i,a}$ and consequently $k_i \cup k_j$ is an algebraic extension of $k_{i,m}$ for every j. Thus we have shown that K^* contains $k_{i,m}$ and is an algebraic extension of $k_{i,m}$. Let k_0 be the prime field and let W_i be the open algebraic variety which is the set of points $c(f(V))$, where f is a non-degenerate projective embedding determined by a divisor in $\mathfrak{A}(X_i)$ (cf. Th. 3). Let (w_i) be a set of quantities such that $k_0((w_i)) = k_i$. Let U_i be a generic variety of $\mathfrak{P}(V, \mathfrak{A}(mX_i))$ over k_i and $k_{i,m}(u_i)$ be the smallest field of definition for U_i containing $k_{i,m}$. There is an isomorphism between V and U_i, say f_i, such that $f_i(\mathfrak{A}(mX_i))$ is the complete total family of U_i-divisors containing hyperplane sections of U_i, and we have clearly $\mathfrak{P}(V, \mathfrak{A}(X_j)) = \mathfrak{P}(U_i, f_i(\mathfrak{A}(X_j)))$. Let α be an automorphism of the universal domain over $k_{i,m}(u_i)$. We have $U_i^\alpha = U_i$, $(f_i(\mathfrak{A}(X_j)))^\alpha = \mathfrak{A}((f_i(X_j))^\alpha)$. Therefore $k_{i,m}((w_j)^\alpha)$ is the smallest field of definition for $\mathfrak{P}(U_i, \mathfrak{A}((f_i(X_j))^\alpha))$. Since $f_i(\mathfrak{A}(mX_i))$ contains a rational divisor over $k_{i,m}(u_i)$, it follows that $f_i(X_j)^\alpha \equiv f_i(X_i) \bmod G_n(V)$ and this shows that $k_{i,m}((w_j)^\alpha)$ coincides with one of the $k_{i,m}((w_j))$. Since $K^* = k_{i,m}((w_1), \cdots, (w_r))$, it follows that K^* is a normal extension of $k_{i,m}$ and $[K^* : k_{i,m}]_s \leqq r$.

Let $k'_{i,m}$ be the largest purely inseparable extension of $k_{i,m}$ in K^*. When t is a positive integer, we have $k_{i,m} \supset k_{i,tm}$ by Lemma 9 and $k'_{i,tm}$ is the smallest subfield of K^* containing $k_{i,tm}$ over which K^* is a separably algebraic extension. Therefore, we have $k'_{i,m} \supset k'_{i,tm}$ and $[K^* : k'_{i,tm}] \leqq r$. From this we see that there is a positive integer t_0 such that whenever $t > t_0$,

we have $k'_{i,t_0m} = k'_{i,tm}$. Let us take m and m' to be prime to the characteristic p (this is certainly possible because of our Theorem 1) and put

$$K = k'_{i,t_0m}.$$

We are going to show that K is the field we are looking for. Let $\mathfrak{P}(V, \mathfrak{A}(Y))$ be a component of $\mathfrak{P}(V, \mathfrak{X}^*)$ and (y) be a set of quantities such that $k_0((y))$ is the smallest field of definition for $\mathfrak{P}(V, \mathfrak{A}(Y))$. Y is contained in $s\mathfrak{N}^*$ for a suitable choice of s by Proposition 4. Let t be a positive integer prime to the characteristic p. Then $b = tmm'$ is prime to p and $\mathfrak{N}(bsX_i) = \mathfrak{N}(bm'Y)$. Moreover, we have $K = k'_{i,bs}$. Let U be a generic variety of $\mathfrak{P}(V, \mathfrak{A}(bsX_i))$ over K, then it is still a generic variety of it over K^* and there is an isomorphism g between V and U such that $g(\mathfrak{A}(bsX_i))$ is the complete total family on U, containing hyperplane sections of it. Let $K(u)$ be the smallest field of definition for U containing K, which is a regular extension of K by Theorem 3. Then since $g(\mathfrak{A}(bsX_i))$ contains a rational divisor over $K(u)$, it is defined over $K(u)$ (cf. Matsusaka [6], Prop. 1).

Assume for a moment that $G_n(U)/G_a(U)$ has a set of representatives in $G_n(U)$ which consists of divisors rational over a separably generated extension of $K(u)$. Then $\mathfrak{N}(bsX_i)$ is covered by r complete total families consisting of numerically effective divisors by our choice of m' and each family is defined over a separably generated extension of $K(u)$ (cf. Matsusaka [6], Prop. 1), and consequently over a separably algebraic extension of $K(u)$. Therefore $\mathfrak{A}(g(bm'Y))$ on U is defined over a separably algebraic extension of $K(u)$ and since b and m' are prime to p, it follows that $\mathfrak{A}(g(Y))$ itself is defined over a separably algebraic extension of $K(u)$ by Lemma 10. This implies that $\mathfrak{P}(U, \mathfrak{A}(g(Y)))$ is defined over a separably algebraic extension of $K(u)$ by Theorem 3; since $\mathfrak{P}(V, \mathfrak{A}(Y)) = \mathfrak{P}(U, \mathfrak{A}(g(Y)))$, our field K satisfies our requirements (i) and (ii). Let us prove now that K is the smallest field having the properties (i) and (ii). Let K' be any field satisfying (i) and (ii). By Lemma 8 and by the definition of K^*, $K^* \cup K'$ is a separably algebraic extension of K'. By (i), there is a component $\mathfrak{P}(V, \mathfrak{A}(Y))$ of $\mathfrak{P}(V, \mathfrak{X}^*)$ such that it is defined over K'. There are two positive integers b and c such that $\mathfrak{A}(bX_i) = \mathfrak{A}(cY)$ and we may take b to be a multiple of t_0m. Then $k_{i,b}$ is the smallest field of definition for $\mathfrak{P}(V, \mathfrak{A}(cY))$, which is a subfield of K' by Lemma 9. By our choice of b, we have $k_{i,b} \subset k'_{i,b} = K$. It follows that K is a purely inseparable extension of $K \cap K'$. Let d be any element of K, then d is purely inseparable over K' and is contained in $K^* \cup K'$. Since $K^* \cup K'$ is separably algebraic over K', d must be in K', which proves that K is a subfield of K'.

Now let us prove that $G_n(U)/G_a(U)$ has a set of representatives in $G_n(U)$ which consists of divisors, rational over a separably generated extension of $K(u)$. First assume that r is prime to p. Put $Z=g(bsX_i)$, then $\mathfrak{N}(Z)$ contains a rational divisor over $K(u)$, i.e. a hyperplane section of U, and is covered by r distinct complete total families. Let $\mathfrak{A}_1,\cdots,\mathfrak{A}_r$ be such and assume that Z is contained in $\mathfrak{A}_1$. Then $\mathfrak{A}_1$ is defined over $K(u)$ since it is our $g(\mathfrak{A}(bsX_i))$. Since $r\mathfrak{A}_1=r\mathfrak{A}_i$, $r\mathfrak{A}_i$ is also defined over $K(u)$ and hence $\mathfrak{A}_i$ is defined over a separably algebraic extension of $K(u)$ by Lemma 10. Therefore $\mathfrak{A}_i$ contains a divisor which is rational over a separably algebraic extension of $K(u)$. Our assertion is proved in this case. Next assume that isomorphisms between V', V'' in $\mathfrak{P}(V,\mathfrak{X}^*)$ are defined over separably generated extensions of any common field of definition for V', V''. Let $U_1,\cdots,U_r$ be independent generic varieties of $\mathfrak{P}(V,\mathfrak{A}(X_1)),\cdots,\mathfrak{P}(V,\mathfrak{A}(X_r))$ over $K^*(u)$. There is a regular extension $K^*(u,w)$ of $K^*(u)$ over which every U_i is defined. Let g, f_i be isomorphisms between V and U, V and U_i respectively. Then $g\cdot f_i^{-1}$ is an isomorphism between U_i and U which is defined over a separably generated extension of $K^*(u,w)$ by our assumption. $\mathfrak{A}(f_i(X_i))$ is defined over $K^*(u,w)$ and since we have $g(X_i)=g\cdot f_i^{-1}(f_i(X_i))$, $\mathfrak{A}(g(X_i))$ is defined over a separably generated extension of $K^*(u,w)$. Hence $\mathfrak{A}(g(X_i))$ contains a divisor which is rational over a separably generated extension of $K^*(u,w)$ and our assertion is proved also in this case.

When $p=0$, the second condition about the field of moduli is automatically satisfied. Therefore, K is the smallest field over which one of components of $\mathfrak{P}(V,\mathfrak{X}^*)$ is defined. Our theorem is thereby proved completely.

Assumptions in our theorem are satisfied when $p=0$, or when V is a polarized abelian variety (cf. Chow [2]). Hence in these cases, there exist fields of moduli.

14. Let U be an algebraic variety and $G=\{f_i\}$, $0\leqq i\leqq r$, be a finite group of automorphisms of U. *The quotient variety of U with respect to G* is, by definition, the following pair (W,ϕ) of a variety W and a rational mapping ϕ of U onto W:

(i) ϕ is defined everywhere on U;

(ii) $\phi=\phi\cdot f_i$ for all i;

(iii) When ϕ' is a rational mapping of U onto a variety W' with $\phi'=\phi'\cdot f_i$ for all i, then $\phi'=\psi\cdot\phi$, where ψ is a rational mapping of W onto W' such that when ϕ' is defined at u' on U, ψ is defined at $\phi(u')$.

Sometimes ϕ is called the *canonical mapping* of U onto W. (W, ϕ) is clearly defined uniquely up to an isomorphism when it exists. When U is a projective variety, the existence has been proved by Serre (cf. J. P. Serre [12], § 3, no. 13). Moreover, when that is so, W can be realized as a projective variety; when U is defined over a field k, the f_i are defined over a separably algebraic extension of k and $f_i^\alpha = f_j \in G$ whenever α is an automorphism of the universal domain over k, then there is a quotient variety (W, ϕ) such that W and ϕ are defined over k. Let now A be a polarized abelian variety, polarized by a set of divisors $\mathfrak{X}$. By Corollary 1, Theorem 6, $\mathfrak{G}(A, \mathfrak{X})$ is a finite group and when k is a field of defintion for A, every automorphism in $\mathfrak{G}(A, \mathfrak{X})$ is defined over a separably algebraic extension of k (cf. Chow [2]). Hence there is a quotient variety (W, ϕ) of A with respect to $\mathfrak{G}(A, \mathfrak{X})$ such that W, ϕ are defined over k and that W is a projective variety. In general, the quotient variety (W, ϕ) of A with respect to $\mathfrak{G}(A, \mathfrak{X})$ shall be called a *generalized Kummer variety* of the polarized abelian variety A. When the ring of endomorphisms of A is isomorphic to the ring of integers, $\mathfrak{G}(A, \mathfrak{X})$ consists of two elements and in this case our generalized Kummer variety coincides with the usual Kummer variety of A. It is easy to see that any generalized Kummer variety is such that its Albanese variety reduces to a point.

THEOREM 8. *Let V be a polarized variety (resp. polarized abelian variety), polarized by a set $\mathfrak{X}$ and assume that there exists the field of moduli. Assume the following:*

(a) *$\mathfrak{G}(V, \mathfrak{X})$ is a finite group;*

(b) *For any pair (U, U') of elements in $\mathfrak{P}(V, \mathfrak{X})$ and for any choice of a common field of definition K' for U and U', isomorphisms between U and U' are defined over separably generated extensions of K'.*

Let K be the field of moduli of V, then to each U in $\mathfrak{P}(V, \mathfrak{X})$, there is a quotient variety (W, ϕ_U) of U with respect to the group of automorphisms of the polarized U with the following properties:

(i) *W is defined over K and is independent of U;*

(ii) *ϕ_U is defined over the smallest field of definition for U containing K;*

(iii) *When U and U' are any two elements in $\mathfrak{P}(V, \mathfrak{X})$ and $f_{U',U}$ is an isomorphism of U to U', then $\phi_{U'} = \phi_U \cdot f_{U',U}^{-1}$.*

(Remark: When V is a polarized abelian variety, (a) and (b) are automatically satisfied (cf. our Cor. 1; Th. 6; Weil [17], Th. 6; Chow [2])).

Proof. Let U_1 and U_2 be elements of $\mathfrak{P}(V, \mathfrak{X})$ and k_1, k_2 be the smallest fields of definitions for U_1, U_2 respectively. Let the (W_i, ϕ_i) be quotient varieties of the U_i with respect to the groups of automorphisms $\mathfrak{G}(U_i)$ of U_i. We shall show that there is an isomorphism $\psi_{2,1}$ of W_1 and W_2 such that $\phi_2 \cdot f_{2,1} = \psi_{2,1} \cdot \phi_1$ for any isomorphism $f_{2,1}$ of U_1 to U_2, and that when k' is a common field of definition for W_1, ϕ_1, W_2 and ϕ_2, $\psi_{2,1}$ is defined over k'. It is easy to see that our two groups $\mathfrak{G}(U_1)$ and $\mathfrak{G}(U_2)$ are isomorphic by the correspondence $f_j^{(1)} \to f_{2,1} \cdot f_j^{(1)} \cdot f_{2,1}^{-1}$, where the $f_j^{(1)}$ are elements of $\mathfrak{G}(U_1)$. Moreover, every isomorphism between U_1 and U_2 can be written as $f_{2,1} \cdot f_j^{(1)}$. This shows in particular that the set of isomorphisms between U_1 and U_2 is finite by (a). Let x be a generic point of U_1 over k'. We claim that the mapping $\psi_{2,1}$ defined by $\phi_1(x) \to \phi_2(f_{2,1}(x))$ satisfies all our requirements. In fact, since $f_{2,1} \cdot f_j^{(1)}$ can be written as $f_i^{(2)} \cdot f_{2,1}$, where $f_i^{(2)}$ is an element of $\mathfrak{G}(U_2)$, we see that

$$\phi_2(f_{2,1}(f_j^{(1)}(x))) = \phi_2(f_i^{(2)}(f_{2,1}(x))) = \phi_2(f_{2,1}(x)).$$

Hence our assertion is proved.

By Theorem 7, there is a component $\mathfrak{A}$ of $\mathfrak{X}^*$ such that $\mathfrak{P}(V, \mathfrak{A})$ is defined over the field of moduli K. By our definition of $\mathfrak{X}^*$, $\mathfrak{A}$ is a complete total family consisting of algebraically effective divisors. Let U_u and U_v be two independent generic elements of $\mathfrak{P}(V, \mathfrak{A})$ over K such that $K(u)$ and $K(v)$ be the smallest fields of definitions for U_u and for U_v containing K (cf. Th. 3). Then $K(u)$ and $K(v)$ are regular extensions of K such that they are linearly disjoint over K. Then there exist quotient varieties $(W_u, \bar{\phi}_u)$, $(W_v, \bar{\phi}_v)$ of U_u, U_v with respect to the groups of automorphisms $\mathfrak{G}(U_u)$, $\mathfrak{G}(U_v)$ of polarized varieties U_u, U_v, defined over $K(u), K(v)$ respectively by our assumptions (a) and (b). By what we have observed, there is an isomorphism $\psi_{u,v}$ of W_u to W_v defined over $K(u, v)$ such that $\bar{\phi}_v \cdot f_{v,u} = \psi_{v,u} \cdot \bar{\phi}_u$, for any isomorphism $f_{v,u}$ between U_u and U_v. Let now u, v and w be three independent generic specializations of u over K. Then we see easily that $\psi_{w,v} \cdot \psi_{v,u} = \psi_{w,u}$. Since U_u, U_v, U_w are projective varieties, we may assume that W_u, W_v, W_w are again projective varieties. Therefore, there is a variety W in a projective space defined over K and the isomorphism h_u between W_u and W defined over $K(u)$ such that $\psi_{v,u} = h_v^{-1} \cdot h_u$ (cf. Weil [21], Th. 5). Put $\phi_u = h_u \cdot \bar{\phi}_u$. Clearly, (W, ϕ_u) is also a quotient variety of U_u with respect to $\mathfrak{G}(U_u)$. Moreover, for any isomorphism $f_{v,u}$ of U_u to U_v, we have $\phi_v \cdot f_{v,u} = \phi_u$.

Let U' be any element of $\mathfrak{P}(V, \mathfrak{X})$ and $(W', \bar{\phi}')$ be a quotient variety of U' with respect to $\mathfrak{G}(U')$. By what we have seen in the beginning of our proof, there is an isomorphism ψ' between W and W' such that, for any isomorphism f_v' of U_v to U', we have $\bar{\phi}' \cdot f_v' = \psi' \cdot \phi_v$. Hence if we put $\phi' = \psi'^{-1} \cdot \bar{\phi}'$, (W, ϕ') is a quotient variety of U' and satisfies (i) and (iii). Let K' be the smallest field of definition for U' containing K. By (b), we may then assume that $(W', \bar{\phi}')$ is defined over K'. By the definition of ϕ', $\phi' = \phi_v \cdot f_v'^{-1}$ for any isomorphism f_v' of U_v and U'. Therefore ϕ' is defined over $K'(v)$ by our assumption (b). Since we can write also $\phi' = \phi_u \cdot f_u'^{-1}$ for any isomorphism f_u' of U_u and U', ϕ' is also defined over $K'(u)$ and this proves that K' is a field of definition for ϕ'. Our theorem is thereby proved.

THE UNIVERSITY OF CHICAGO, AND
NORTHWESTERN UNIVERSITY.

REFERENCES.

[1] N. Bourbaki, "Théorie des ensembles," *Actualités Scientifiques et Industrielles*, No. 846 (1951).

[2] W. L. Chow, "Abelian varieties over function fields," *Transactions of the American Mathematical Society*, vol. 78 (1955), pp. 253-275.

[3] ———, "Abstract theory of the Picard and Albanese varieties," to appear.

[4] ——— and B. L. van der Waerden, "Zur Algebraischen Geometrie IX," *Mathematische Annalen*, vol. 113 (1937), pp. 692-794.

[5] T. Matsusaka, "On the algebraic construction of the Picard variety," *Japanese Journal of Mathematics*, vol. 21 (1951), pp. 217-235.

[6] ———, "On the algebraic construction of the Picard variety II," *ibid.*, vol. 22 (1952), pp. 51-62.

[7] ———, "On the algebraic families of positive divisors," *Journal of the Mathematical Society of Japan*, vol. 5 (1953), pp. 115-136.

[8] ———, "Some theorems on abelian varieties," *Natural Science Report*, Ochanomizu University, vol. 4 (1953), pp. 22-25.

[9] ———, "The criteria for algebraic equivalence and the torsion group," *American Journal of Mathematics*, vol. 79 (1957), pp. 53-66.

[10] A. Néron, "Problèmes arithmétiques et géométriques rattachés a la notion du rang d'une courbe algébriques dans un corps," *Bulletin de la Société Mathématique de France*, vol. 80 (1952), pp. 101-166.

[11] ——— and P. Samuel, "La variété de Picard d'une variété Normale," *Annales de l'institut Fourier*, vol. 4 (1952), pp. 1-30.

(Remark: When V is a polarized abelian variety, (a) and (b) are automatically satisfied (cf. our Cor. 1; Th. 6; Weil [17], Th. 6; Chow [2])).

Proof. Let U_1 and U_2 be elements of $\mathfrak{P}(V, \mathfrak{X})$ and k_1, k_2 be the smallest fields of definitions for U_1, U_2 respectively. Let the (W_i, ϕ_i) be quotient varieties of the U_i with respect to the groups of automorphisms $\mathfrak{G}(U_i)$ of U_i. We shall show that there is an isomorphism $\psi_{2,1}$ of W_1 and W_2 such that $\phi_2 \cdot f_{2,1} = \psi_{2,1} \cdot \phi_1$ for any isomorphism $f_{2,1}$ of U_1 to U_2, and that when k' is a common field of definition for W_1, ϕ_1, W_2 and ϕ_2, $\psi_{2,1}$ is defined over k'. It is easy to see that our two groups $\mathfrak{G}(U_1)$ and $\mathfrak{G}(U_2)$ are isomorphic by the correspondence $f_j^{(1)} \to f_{2,1} \cdot f_j^{(1)} \cdot f_{2,1}^{-1}$, where the $f_j^{(1)}$ are elements of $\mathfrak{G}(U_1)$. Moreover, every isomorphism between U_1 and U_2 can be written as $f_{2,1} \cdot f_j^{(1)}$. This shows in particular that the set of isomorphisms between U_1 and U_2 is finite by (a). Let x be a generic point of U_1 over k'. We claim that the mapping $\psi_{2,1}$ defined by $\phi_1(x) \to \phi_2(f_{2,1}(x))$ satisfies all our requirements. In fact, since $f_{2,1} \cdot f_j^{(1)}$ can be written as $f_i^{(2)} \cdot f_{2,1}$, where $f_i^{(2)}$ is an element of $\mathfrak{G}(U_2)$, we see that

$$\phi_2(f_{2,1}(f_j^{(1)}(x))) = \phi_2(f_i^{(2)}(f_{2,1}(x))) = \phi_2(f_{2,1}(x)).$$

Hence our assertion is proved.

By Theorem 7, there is a component $\mathfrak{A}$ of $\mathfrak{X}^*$ such that $\mathfrak{P}(V, \mathfrak{A})$ is defined over the field of moduli K. By our definition of $\mathfrak{X}^*$, $\mathfrak{A}$ is a complete total family consisting of algebraically effective divisors. Let U_u and U_v be two independent generic elements of $\mathfrak{P}(V, \mathfrak{A})$ over K such that $K(u)$ and $K(v)$ be the smallest fields of definitions for U_u and for U_v containing K (cf. Th. 3). Then $K(u)$ and $K(v)$ are regular extensions of K such that they are linearly disjoint over K. Then there exist quotient varieties $(W_u, \bar{\phi}_u)$, $(W_v, \bar{\phi}_v)$ of U_u, U_v with respect to the groups of automorphisms $\mathfrak{G}(U_u)$, $\mathfrak{G}(U_v)$ of polarized varieties U_u, U_v, defined over $K(u), K(v)$ respectively by our assumptions (a) and (b). By what we have observed, there is an isomorphism $\psi_{u,v}$ of W_u to W_v defined over $K(u, v)$ such that $\bar{\phi}_v \cdot f_{v,u} = \psi_{v,u} \cdot \bar{\phi}_u$ for any isomorphism $f_{v,u}$ between U_u and U_v. Let now u, v and w be three independent generic specializations of u over K. Then we see easily that $\psi_{w,v} \cdot \psi_{v,u} = \psi_{w,u}$. Since U_u, U_v, U_w are projective varieties, we may assume that W_u, W_v, W_w are again projective varieties. Therefore, there is a variety W in a projective space defined over K and the isomorphism h_u between W_u and W defined over $K(u)$ such that $\psi_{v,u} = h_v^{-1} \cdot h_u$ (cf. Weil [21], Th. 5). Put $\phi_u = h_u \cdot \bar{\phi}_u$. Clearly, (W, ϕ_u) is also a quotient variety of U_u with respect to $\mathfrak{G}(U_u)$. Moreover, for any isomorphism $f_{v,u}$ of U_u to U_v, we have $\phi_v \cdot f_{v,u} = \phi_u$.

Let U' be any element of $\mathfrak{P}(V, \mathfrak{X})$ and $(W', \bar{\phi}')$ be a quotient variety of U' with respect to $\mathfrak{G}(U')$. By what we have seen in the beginning of our proof, there is an isomorphism ψ' between W and W' such that, for any isomorphism f_v' of U_v to U', we have $\bar{\phi}' \cdot f_v' = \psi' \cdot \phi_v$. Hence if we put $\phi' = \psi'^{-1} \cdot \bar{\phi}'$, (W, ϕ') is a quotient variety of U' and satisfies (i) and (iii). Let K' be the smallest field of definition for U' containing K. By (b), we may then assume that $(W', \bar{\phi}')$ is defined over K'. By the definition of ϕ', $\phi' = \phi_v \cdot f_v'^{-1}$ for any isomorphism f_v' of U_v and U'. Therefore ϕ' is defined over $K'(v)$ by our assumption (b). Since we can write also $\phi' = \phi_u \cdot f_u'^{-1}$ for any isomorphism f_u' of U_u and U', ϕ' is also defined over $K'(u)$ and this proves that K' is a field of definition for ϕ'. Our theorem is thereby proved.

THE UNIVERSITY OF CHICAGO, AND
NORTHWESTERN UNIVERSITY.

REFERENCES.

[1] N. Bourbaki, "Théorie des ensembles," *Actualités Scientifiques et Industrielles*, No. 846 (1951).

[2] W. L. Chow, "Abelian varieties over function fields," *Transactions of the American Mathematical Society*, vol. 78 (1955), pp. 253-275.

[3] ———, "Abstract theory of the Picard and Albanese varieties," to appear.

[4] ——— and B. L. van der Waerden, "Zur Algebraischen Geometrie IX," *Mathematische Annalen*, vol. 113 (1937), pp. 692-794.

[5] T. Matsusaka, "On the algebraic construction of the Picard variety," *Japanese Journal of Mathematics*, vol. 21 (1951), pp. 217-235.

[6] ———, "On the algebraic construction of the Picard variety II," *ibid.*, vol. 22 (1952), pp. 51-62.

[7] ———, "On the algebraic families of positive divisors," *Journal of the Mathematical Society of Japan*, vol. 5 (1953), pp. 115-136.

[8] ———, "Some theorems on abelian varieties," *Natural Science Report*, Ochanomizu University, vol. 4 (1953), pp. 22-25.

[9] ———, "The criteria for algebraic equivalence and the torsion group," *American Journal of Mathematics*, vol. 79 (1957), pp. 53-66.

[10] A. Néron, "Problèmes arithmétiques et géométriques rattachés a la notion du rang d'une courbe algébriques dans un corps," *Bulletin de la Société Mathématique de France*, vol. 80 (1952), pp. 101-166.

[11] ——— and P. Samuel, "La variété de Picard d'une variété Normale," *Annales de l'institut Fourier*, vol. 4 (1952), pp. 1-30.

[12] J. P. Serre, "Sur la topologie des variétés algébriques en caractéristique p," *Symposium de topologie algébrique*, Mexico, 1956, to appear.

[13] G. Shimura, "Reduction of algebraic varieties with respect to a discrete valuation of the basic field," *American Journal of Mathematics*, vol. 77 (1955), pp. 134-176.

[14] C. L. Siegel, "Lectures on the analytic theory of quadratic forms," Princeton lecture note, revised edition (1949).

[15] B. L. van der Waerden, *Einführung in die algebraische Geometrie*, Berlin (1939).

[16] A. Weil, *Foundations of Algebraic Geometry*, American Mathematical Society, Colloquium Publication, vol. 29 (1946).

[17] ———, "Variétés Abéliennes et Courbes Algébriques," *Actualités Scientifiques et Industrielles*, No. 1064 (1948).

[18] ———, "Sur les critères d'équivalence en géométrie algébrique," *Mathematische Annalen*, vol. 128 (1954), pp. 95-127.

[19] ———, "Projective embeddings of abelian varieties," Algebraic Geometry and Topology (A symposium in honor of S. Lefschetz), pp. 177-181.

[20] ———, "On algebraic groups of transformations," *American Journal of Mathematics*, vol. 77 (1955), pp. 355-391.

[21] ———, "The field of definition of a variety," *ibid.*, vol. 78 (1956), pp. 509-524.

[22] ———, "On the theory of complex multiplication," *Proceedings of the International symposium on Algebraic Number Theory* (1955), pp. 9-22.

[23] O. Zariski, "Foundations of a general theory of birational correspondences," *Transactions of the American Mathematical Society*, vol. 53 (1943), pp. 490-542.

CONFORMAL INVARIANTS AND PRIME ENDS.*[1]

By Ernest C. Schlesinger.

Introduction. From the point of view of conformal mapping, it is unsatisfactory to consider the individual points of the boundary of a simply connected region as the primitive constituents of this boundary. When such a region is mapped conformally onto the unit disk, in accordance with the Riemann mapping theorem, the points of the unit circumference correspond, indeed, to the prime ends of the region. These prime ends are here considered as equivalence classes of sequences of points of the region under a relation P^*. The principal tool we employ to introduce this relation is extremal length, a conformally invariant quantity associated with a family of curves. Its definition was first given by Ahlfors and Beurling ([2], [3]).

The term "prime end" originated with Carathéodory [4], who initiated the systematic study of the structure of the boundary of a simply connected region. His approach was topological, and it dealt with concepts—subregions, crosscuts, etc.—which are defined with reference to the given region. It is easily seen that with Carathéodory's definition the prime ends of the unit disk correspond to points on the circumference in the sense that an equivalence class is associated with each boundary point and that two sequences belong to the same class if and only if they converge to the same point of the circumference. The problem that arises under his approach, however, and this is solved by one of Carathéodory's fundamental theorems, is that one must show that prime ends are preserved under conformal mapping.

Lindelöf [6] circumvented this difficulty by defining prime ends by reference to the conformal map of the disk onto the region; namely in terms of the set of indetermination or cluster set. However, his method does not obviate an explicit analysis of the topological situation in the region itself. Lindelöf also obtained the following result, which is related to the classification of prime ends.

Carathéodory had distinguished two classes of points, principal and subsidiary (see Section 2.7, below), on the point set of a prime end, i.e. on the

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set of accumulation points of all sequences belonging to the prime end. Lindelöf showed that if a sequence of points of the disk converges to a boundary point in an angle (non-tangentially) then the image sequence has only principal points of the corresponding prime end for accumulation points. This is a precise way of stating that the principal points form a kind of nucleus for the point set of the prime end.

Two other schemes for defining prime ends deserve brief mention. Mazurkiewicz [7] introduced a metric, $\rho_\pi(z_1, z_2)$, equivalent to the euclidean metric in the region in the sense that $\rho_\pi(z_j, z_0) \to 0$ if and only if $| z_j - z_0 | \to 0$ for any sequence $\{z_j\}_1^\infty$ of points of the region. The boundary of the region with respect to ρ_π (i.e. the complement of the region with respect to its ρ_π-completion) is a space which can be identified with the set of prime ends of Carathéodory.

Ursell and Young [9] have used the notion of an equivalence class of paths, which converge to the boundary of the region, to introduce the prime ends of the region.

The purpose of the present paper is to define prime ends once more. We use extremal length, and thus, we are able to give our definition in terms of constructs related to the region itself, while we obtain the conformally invariant nature of our prime ends immediately, since extremal length is a conformally invariant quantity. Finally, we state and prove in a conformally invariant form the theorem of Lindelöf, relating to approach in an angle, to which we referred above. For this purpose we attach a conformal invariant (an extremal length) to each point of a sequence which converges to a prime end. Our condition is then formulated in terms of the boundedness of the sequence of conformal invariants.

The author takes this opportunity to express his particular gratitude to Professor Lars V. Ahlfors of Harvard University who suggested the problem and whose encouragement and guidance at every stage made the completion of the study possible.

I. Extremal Length.

1.0 Summary. Our principal tool is extremal length, a conformally invariant quantity associated with a family of curves. In this section, we first make precise the special meaning of some of the terms used in the sequel. Then we give the definition of extremal length, and state, partly without proof, those of its properties which are needed for our investigation.

1.1 All our point sets are assumed to lie on the Riemann sphere, which we denote by Σ. The spherical distance, $\psi(z, w)$, of the points with coordinates z and w, is used throughout.

A region, as always, is an open connected subset of Σ; it is simply connected if its complement with respect to Σ is connected. The closure and boundary of a region Ω are denoted by $\bar{\Omega}$ and $bd\,\Omega$, respectively. A configuration consists of a region and a set of distinguished points of the closure of the region. This set may contain interior points, isolated boundary points, or sub-continua of the boundary of the region.

Let Ω be a region. An arc of Ω shall be a continuous map, $z = z_\gamma(t)$, of an (open, half-open, or closed) interval into Ω. We denote the map and the point set of the arc, i.e. the subset of the region Ω onto which the interval is mapped, by the same symbol γ; when we want to emphasize that we are dealing with the point set, we shall occasionally use (γ). A closed arc of Ω is the continuous map of a closed interval into Ω for which the images of the initial and terminal points coincide. An arc γ is simple, if $z_\gamma(t_1) = z_\gamma(t_2)$ only if $t_1 = t_2$, or (in case of a closed arc) if t_1 and t_2 are endpoints of the parameter interval.

We shall also consider certain finite collections of arcs of Ω; such a collection of arcs we shall call a *generalized curve.*

An arc γ of Ω will be called a *crosscut of* Ω, provided its point set, (γ), is the homeomorph of an open interval, and $\Omega - (\gamma)$ has precisely two components.[2] When we consider, in the sense of the preceding paragraph, generalized curves whose constituent arcs are crosscuts, we shall insist that the corresponding point set be connected. Thus we have the following definition.

DEFINITION 1. *A generalized crosscut of* Ω *is a finite collection* τ *of crosscuts* $\gamma_1, \gamma_2, \cdots, \gamma_n$ *of* Ω, *such that the point set* $(\tau) = \bigcup_{j=1}^{n} (\gamma_j)$ *is connected.*

1.2 Let Ω be a region, $\{\gamma\}$ a family of generalized curves of Ω, and let $\rho(z)$ be a non-negative real-valued function defined on Ω. If $\rho^2(z)$ is

[2] This is a modification of the classical definition, according to which the point set of a crosscut is the homeomorph of an open interval with the additional property that this homeomorphism is the restriction of a continuous map of the closure of the interval into the closure of the region, and that the endpoints of the interval correspond to points on the boundary of the region. It is well known that these classical crosscuts have the separating property demanded above ([5]; p. 106). However, this formulation of the notion of a crosscut does not yield a conformally invariant concept.

integrable over Ω, we set $A_\Omega(\rho) = A(\rho) = \iint_\Omega \rho^2(z)\,dxdy$ $(z = x + iy)$, and otherwise, $A_\Omega(\rho) = \infty$.

Let γ be a curve of the family $\{\gamma\}$. If there is a finite or countable sequence $\{\gamma_i\}$ of disjoint rectifiable arcs, which are parameterized by their arc lengths (i.e. the arc γ_i is given by $z = z_i(s_i)$, where $0 \leqq s_i \leqq l_i$, and l_i is the length of γ_i), such that $(\gamma) = \bigcup_1^\infty (\gamma_i)$, and if the integrals appearing in the following sum exist in the sense of Lebesgue, we set

$$l_\rho(\gamma) = \sum_{i=1}^{\infty} \int_{\gamma_i} \rho(z_i(s_i))\,ds_i.$$

If γ cannot be represented in the above fashion, we set $l_\rho(\gamma) = \infty$. Finally, we introduce the quantity

$$L(\rho) = L_{\{\gamma\}}(\rho) = \inf_{\gamma \in \{\gamma\}} l_\rho(\gamma),$$

where we agree that $L_{\{\gamma\}}(\rho) = \infty$ in case the family $\{\gamma\}$ is empty.

1.3 Extremal length. The function $\rho(z)$ shall be called *admissible* (with respect to Ω and $\{\gamma\}$) if $A(\rho)$ and $L(\rho)$ are not both zero or both infinite.

DEFINITION 2. ([2], [3], [8]) *The quantity*

$$\lambda_\Omega(\gamma) = \lambda(\gamma) = \sup L^2(\rho)/A(\rho),$$

where the supremum is taken over all admissible functions $\rho(z)$, is called the extremal length of the family of curves $\{\gamma\}$ with respect to the region Ω.

The following observations are found in the literature. (For example: [1], [8].)

(a) Extremal length is a conformal invariant.

(b) The extremal length $\lambda_\Omega(\gamma)$ depends only on the carrier of the family of curves $\{\gamma\}$ and not on the remainder of the region Ω.

(c) Since the ratio $L^2(\rho)/A(\rho)$ is homogeneous of degree 0 in ρ, we can introduce certain normalizations in the class of admissible functions:

(i) Under the additional restriction that $\lambda(\gamma) < \infty$, we have $\lambda(\gamma) = \sup L_{\{\gamma\}}{}^2(\rho)$, where the supremum is now taken over all admissible functions ρ for which $A(\rho) = 1$.

(ii) If $\lambda(\gamma) > 0$, we have $\lambda^{-1}(\gamma) = \inf A(\rho)$, with the infimum taken over admissible ρ for which $L_{\{\gamma\}}(\rho) \geqq 1$.

(iii) In case $0 < \lambda(\gamma) < \infty$, we may take $\lambda(\gamma) = \sup L_{\{\gamma\}}(\rho)$, the supremum being taken over all admissible functions ρ which satisfy the condition $L_{\{\gamma\}}(\rho) = A(\rho)$.

1.4 The comparison principle. The following lemma and its corollary allow us to give simple estimates for the extremal length. The proof is included as a typical one.

LEMMA 1. *Let $\{\gamma\}$ and $\{\gamma'\}$ be two families of generalized curves of a region Ω such that each $\gamma \in \{\gamma\}$ contains a $\gamma' \in \{\gamma'\}$. Then $\lambda(\gamma) \geqq \lambda(\gamma')$.*

Proof. Let $\epsilon > 0$ be given and let $\rho'(z)$ be admissible for the family $\{\gamma'\}$, subject to the condition that

$$L_{\{\gamma'\}}{}^2(\rho')/A(\rho') > \lambda(\gamma') - \epsilon. \tag{1}$$

We define a function $\rho_0(z)$ in Ω by

$$\rho_0(z) = \rho'(z) \text{ if } z \text{ belongs to some } \gamma', \quad \rho_0(z) = 0 \text{ otherwise.}$$

For $\gamma \in \{\gamma\}$, let γ' be one of the elements of $\{\gamma'\}$ contained in γ. Then

$$l_{\rho_0}(\gamma) \geqq L_{\{\gamma'\}}(\rho'), \tag{2}$$

since either $l_{\rho_0}(\gamma) = \infty$, or

$$l_{\rho_0}(\gamma) = \int_\gamma \rho_0 \, ds \geqq \int_{\gamma'} \rho' \, ds = l_{\rho'}(\gamma') \geqq L_{\{\gamma'\}}(\rho').$$

Formula (2) holds for each $\gamma \in \{\gamma\}$, so that

$$L_{\{\gamma\}}(\rho_0) \geqq L_{\{\gamma'\}}(\rho'). \tag{3}$$

Also, it is clear that

$$A(\rho_0) \leqq A(\rho'). \tag{4}$$

We must show that ρ_0 is admissible for the family $\{\gamma\}$. Since the lemma holds trivially if $\lambda(\gamma') = 0$, we may assume that $A(\rho') < \infty$, and a fortiori $A(\rho_0) < \infty$. If $A(\rho_0) = 0$, then either (i) $A(\rho') = 0$, or (ii) $A(\rho') > 0$. In case (i), $L_{\{\gamma'\}}(\rho') > 0$, since ρ' is admissible for $\{\gamma'\}$, while in case (ii), we choose ϵ so that $0 < \epsilon < \lambda(\gamma')$ in (1), and we conclude from that formula that $L_{\{\gamma'\}}(\rho') > 0$. In either case, therefore, $A(\rho_0) = 0$ implies, by (3), that $L_{\{\gamma\}}(\rho_0) > 0$; hence, ρ_0 is admissible for $\{\gamma\}$. From formulas (3) and (4), we obtain

$$L_{\{\gamma\}}{}^2(\rho_0)/A(\rho_0) \geqq L_{\{\gamma'\}}{}^2(\rho')/A(\rho'),$$

and therefore, by (1), $\lambda(\gamma) > \lambda(\gamma') - \epsilon$. This holds for each positive ϵ, and consequently, $\lambda(\gamma) \geqq \lambda(\gamma')$.

COROLLARY. *In particular, if* $\{\gamma\}$ *is contained in* $\{\gamma'\}$, *then* $\lambda(\gamma) \geqq \lambda(\gamma')$.

1.5 For the following two configurations we can compute extremal lengths explicitly. (For proofs see [1] or [8].)

(a) Let Ω be the interior of a rectangle with sides of lengths a and b, and let $\{\gamma\}$ be the family of crosscuts of Ω which connect the two sides of length a. Then $\lambda(\gamma) = b/a$.

(b) Let Ω be the interior of the annulus formed by two concentric circles of radii r and $R (r < R)$. The family $\{\gamma\}$ of crosscuts of Ω which connect the two circles has extremal length $\lambda(\gamma) = \frac{1}{2\pi} \log(R/r)$, while the family $\{\gamma^*\}$ of simple closed arcs of Ω which separate the two boundary components has extremal length $\lambda(\gamma^*) = 2\pi/\log(R/r)$.

1.6 The next result allows us to conclude that if the union of two families of curves has vanishing extremal length, the same is true of at least one of these two families.

LEMMA 2. *If* $\{\gamma'\}$ *and* $\{\gamma''\}$ *are two non-empty families of generalized curves of a region* Ω, *and* $\{\gamma^0\}$ *is the union of these two families, then* $\lambda(\gamma') > 0$ *and* $\lambda(\gamma^0) = 0$ *imply that* $\lambda(\gamma'') = 0$.

Proof. [8; Satz 5] Assume $\lambda(\gamma'') > 0$, and let ρ' and ρ'' be admissible for the families $\{\gamma'\}$ and $\{\gamma''\}$, respectively; we normalize by the conditions that $L_{\{\gamma'\}}(\rho') \geqq 1$ and $L_{\{\gamma''\}}(\rho'') \geqq 1$. Set $\rho_0(z) = \max\{\rho'(z), \rho''(z)\}$. It is clear that $L_{\{\gamma^0\}}(\rho_0) \geqq 1$, and $A(\rho_0) \leqq A(\rho') + A(\rho'')$. Hence, the function ρ_0 is admissible for the family $\{\gamma^0\}$, and

$$\lambda^{-1}(\gamma^0) \leqq \inf A(\rho_0) \leqq \inf A(\rho') + \inf A(\rho''),$$

where the first infimum is taken for all functions ρ_0 of the above form and the last two infima are taken for all admissible ρ' and $''\rho$. Consequently, $\lambda^{-1}(\gamma^0) \leqq \lambda^{-1}(\gamma') + \lambda^{-1}(\gamma'')$, and this contradicts the assumption that $\lambda(\gamma^0) = 0$.

1.7 The material of the present section is required in the third part of this paper. The proofs are simple applications of the comparison principle (Lemma 1) and Section 1.5 (a).

DEFINITION 3. ([2]) *Let* E_1 *and* E_2 *be closed disjoint subsets of the boundary of a region* Ω. *The extremal length of the family of crosscuts of*

Ω which have one endpoint on E_1 and the other on E_2 is called the extremal distance of E_1 and E_2 with respect to Ω, and it is denoted by $\lambda_\Omega(E_1, E_2)$.

In Section 1.5 (a), we saw that the extremal distance of the vertical sides of a rectangle of dimensions $a \times b$ is a/b; the extremal distance of the horizontal sides is b/a. We note that the product of these two extremal distances is 1.

We are concerned with the following configuration embedded in the unit disk K. Let E_1 be the right semicircular arc, $(+i, +1, -i)$, of the circumference of K, and let E_2 be the segment of the real axis from -1 to the point a $(-1 < a < +1)$. The upper semi-circular region is denoted by S.

LEMMA 3. *The extremal distance of $\bar{S}E_1$ and $\bar{S}E_2$ with respect to S is twice that of E_1 and E_2 with respect to $K - E_2$; in symbols: $\lambda_S(\bar{S}E_1, \bar{S}E_2) = 2\lambda_{K-E_2}(E_1, E_2)$.*

Proof. It is possible to map $K - E_2$ conformally and one-to-one onto a rectangle with horizontal and vertical sides of lengths u and v in such a way that E_1 corresponds to one and E_2 (traversed twice) to the other horizontal side. By conformal invariance, $\lambda_{K-E_2}(E_1, E_2) = v/u$. If the above rectangle is bisected by a vertical line, its right half will correspond to S under the inverse of the above map, and we have

$$\lambda_S(\bar{S}E_1, \bar{S}E_2) = v/(u/2) = 2\lambda_{K-E_2}(E_1, E_2).$$

LEMMA 4. *If E_3 is the left boundary arc of K, and E_4 is the segment of the real axis from the point a to $+1$, then*

$$\lambda_{K-E_4}(E_3, E_4) \cdot \lambda_{K-E_2}(E_1, E_2) = \tfrac{1}{4}.$$

Proof. This follows immediately, for by the preceding lemma we also have $\lambda_S(\bar{S}E_3, \bar{S}E_4) = 2\lambda_{K-E_4}(E_3, E_4)$, and in view of an earlier remark, $\lambda_S(\bar{S}E_3, SE_4) \cdot \lambda_S(\bar{S}E_1, SE_2) = 1$.

THEOREM 1. *The extremal length of the family $\{\gamma\}$ of crosscuts of K with endpoints on E_1 which separate the point a from E_3 is four times the extremal distance of E_1 and E_2 with respect to $K - E_2$.*

Proof. We first show

$$\lambda_K(\gamma) \leqq 2\lambda_S(\bar{S}E_1, \bar{S}E_2). \tag{1}$$

Let $\rho(z)$ be a function which is admissible for $\{\gamma\}$; we introduce the symmetrized function $\hat{\rho} = \rho + \bar{\rho}$, where $\bar{\rho}(z) = \rho(\bar{z})$, and $\bar{z}$ is the conjugate

of z. Then $A_S(\hat{\rho}) \leqq 2A_K(\rho)$, since $A_S(\hat{\rho}) = \frac{1}{2}A_K(\hat{\rho})$ by symmetry, and an elementary estimate (Minkowski's inequality), together with the observation that $A_K(\rho) = A_K(\bar{\rho})$, shows $A_K(\hat{\rho}) \leqq 4A_K(\rho)$.

By $\{\delta\}$ we denote the family of crosscuts of S with endpoints on $\bar{S}E_1$ and $\bar{S}E_2$, respectively. If δ belongs to this family, it, together with $\bar{\delta}$, its reflection in the real axis, forms a crosscut of K which separates a from E_3 and has endpoints on E_1; in short, $\delta \cup \bar{\delta} = \gamma \in \{\gamma\}$. We have

$$L_{\{\gamma\}}(\rho) \leqq l_\rho(\gamma) = l_\rho(\delta \cup \bar{\delta}) = l_{\hat{\rho}}(\delta),$$

and hence, $L_{\{\gamma\}}(\rho) \leqq L_{\{\delta\}}(\hat{\rho})$, so that, since it is immediately verified that $\hat{\rho}$ is admissible for $\{\delta\}$ and S,

$$L_{\{\gamma\}}{}^2(\rho)/A_K(\rho) \leqq 2L_{\{\delta\}}{}^2(\hat{\rho})/A_S(\hat{\rho}) \leqq 2\lambda_S(\delta).$$

This relation holds for each ρ admissible for $\{\gamma\}$ and K, and thus we have formula (1).

Next, we have

$$\lambda_K(\gamma) \geqq 4\lambda_{K-E_2}(E_1, E_2). \tag{2}$$

Namely, if $\{\delta'\}$ is the family of crosscuts of $K - E_2$ with endpoints on E_1 and E_2, respectively, we let ϵ be a positive number, and we consider a function $\rho(z)$, which is admissible for $\{\delta'\}$ and such that

$$L_{\{\delta'\}}{}^2(\rho)/A_{K-E_2}(\rho) > \lambda_{K-E_2}(\delta') - \epsilon/4.$$

Then we define $\rho_0(z)$ for $z \in K$ by placing

$$\rho_0(z) = \rho(z) \text{ if there is a } \delta' \text{ in } \{\delta'\} \text{ for which } z \in \delta',$$

and $\rho_0(z) = 0$ otherwise. Each $\gamma \in \{\gamma\}$ contains two arcs, δ'_1 and δ'_2, of the family $\{\delta'\}$. Consequently,

$$l_{\rho_0}(\gamma) \geqq l_{\rho_0}(\delta'_1) + l_{\rho_0}(\delta'_2) \geqq 2L_{\{\delta'\}}(\rho),$$

and hence, $L_{\{\gamma\}}(\rho_0) \geqq 2L_{\{\delta'\}}(\rho)$; also $A_K(\rho_0) \leqq A_{K-E_2}(\rho)$. From this it follows that

$$\lambda_K(\gamma) \geqq L_{\{\gamma\}}{}^2(\rho_0)/A_K(\rho_0) \geqq 4L_{\{\delta'\}}{}^2(\rho)/A_{K-E_2}(\rho) > 4\lambda_{K-E_2}(\delta') - \epsilon.$$

This holds for each positive ϵ, so that (2) is established. Formulas (1) and (2) together with Lemma 3, now yield the theorem.

II. Property *P* and Relation *P**. Prime Ends.

2.0 Summary. We consider sequences of points of a simply connected region on the Riemann sphere. A P-sequence is one for which the extremal length of a certain associated family of crosscuts is 0. We also introduce Q-sequences by a similar condition relating to a family of generalized crosscuts, and we show that $\{a_n\}$ is a Q-sequence if and only if it is also a P-sequence. Two sequences form a P^*-pair provided their join, the mixed sequence obtained from them, is a P-sequence. Then P^* is an equivalence relation defined on the P-sequences of the region. (It is the proof of the transitivity of the relation P^* which requires the consideration of Q-sequences.)

We shall see that the P^* equivalence classes of a circular disk are in one-to-one correspondence with the points of the circumference, and, in general, for a simply connected region, the P^* classes consist precisely of those P-sequences which converge to prime ends of the region in the sense of Carathédory. We may therefore identify these prime ends with our P^* equivalence clases. Since P-sequences and P^*-pairs are, by definition, conformally invariant concepts, we obtain an immediate proof of Carathédory's fundamental theorems.

2.1 Let Ω be a simply connected region, and let A and B be two sets contained in Ω. Then we say that the generalized crosscut τ of Ω separates A and B with respect to Ω, if $A \cup B \subset \{\Omega - (\tau)\}$, and if the set of components of $\Omega - (\tau)$ containing points of A is disjoint from the set of components containing points of B.

To formulate our principal definition, we consider a sequence $\{a_n\}$ of points of the region Ω, together with a closed subset E of Ω which has the property that a_n belongs to the set $\Omega - E$, except for at most a finite number of subscripts. With this configuration we associate two families of curves:

$\{\gamma\}$, the set of all crosscuts of Ω which separate E from almost all[3] a_n, and

$\{\tau\}$, the set of all generalized crosscuts of Ω each of which separates E from almost all a_n.

DEFINITION 4. *The sequence $\{a_n\}$ has Property P with respect to the set E, provided the associated family $\{\gamma\}$ has extremal length $\lambda(\gamma) = 0$.*

DEFINITION 4a. *The sequence $\{a_n\}$ has Property Q with respect to E, provided the associated family $\{\tau\}$ has extremal length $\lambda(\tau) = 0$.*

[3] As is customary, "almost all a_n" means "all a_n with at most a finite number of exceptions."

Remarks. 1. A sequence with Property P will be called a P-sequence, and similarly, one with Property Q, a Q-sequence.

2. The purpose of the family $\{\tau\}$ and of Property Q is to facilitate the proof of the transitivity of our relation P^* (see Definition 6 and Theorem 4 below).

3. Since $\{\gamma\} \subset \{\tau\}$, $\lambda(\gamma) = 0$ implies $\lambda(\tau) = 0$, by the comparison principle. We shall see in the sequel that it is true as well, that if $\lambda(\tau) = 0$ then $\lambda(\gamma) = 0$; hence the two properties, P and Q, are equivalent.

4. In Theorem 2, we shall see that Property P is independent of the set E; hence, it suffices to speak of P-sequences. When no explicit reference to a set E is made, we assume that there is a fixed closed subset E of Ω with respect to which a given sequence of points of the region either has or fails to have Property P.

LEMMA 5. *If $\{a_n\}$ is a P-sequence of Ω (with respect to a set E), then any subsequence is also a P-sequence.*

Proof. Clearly, almost all points of the subsequence belong to $\Omega - E$, and the family $\{\gamma'\}$, of crosscuts of Ω separating almost all points of the subsequence from E, contains the family $\{\gamma\}$. Hence, $\lambda(\gamma') \leqq \lambda(\gamma) = 0$, and the lemma is established.

LEMMA 6. *All accumulation points of the P-sequence $\{a_n\}$ lie on the boundary of the region Ω.*

Proof. Let $\alpha \in \Omega$ be an accumulation point of the sequence. Then $\alpha \notin E$, since otherwise $\{\gamma\}$ would be empty, and $\lambda(\gamma) = \infty$. Therefore, the open set $\Omega - E$ contains a closed disk Δ with center α such that $a_n \in \Delta$ for an infinite number of indices. If Π is a polynomial arc contained in Ω which connects α to a point β of E, then every crosscut γ of the family $\{\gamma\}$ intersects the continuum $\kappa = \Pi \cup \Delta$. Now, κ has positive spherical distance δ from the boundary. Let $\rho^*(z) = \psi(z, \infty)$, where $\psi(z, w)$ is the spherical distance. Then, for each $\gamma \in \{\gamma\}$, $l_{\rho^*}(\gamma) \geqq 2\delta$, and hence, $L_{\{\gamma\}}(\rho^*) \geqq 2\delta$. Since $0 < A(\rho^*) < \pi$, ρ^* is admissible, and we have

$$\lambda(\gamma) \geqq L^2(\rho^*)/A(\rho^*) \geqq (4/\pi)\delta^2 > 0.$$

This contradiction shows that α cannot be an interior point of the region.

2.2 THEOREM 2. *Let E' be any non-empty closed subset of Ω. Then, if $\{a_n\}$ has property P with respect to E, it also has property P with respect to the set E'.*

Proof. First of all, we note that almost all points a_n belong to $\Omega - E'$, for otherwise the sequence $\{a_n\}$ would have an accumulation point in E', contradicting Lemma 6. We denote by $\{\gamma'\}$ the family of crosscuts of Ω each of which separates E' from almost all a_n, and we show that if $\lambda(\gamma) = 0$ then $\lambda(\gamma') = 0$. Two special cases must be dealt with separately.

(a) If $E' \subset E$, every γ is also a γ'; the comparison principle (Corollary of Lemma 1) yields $\lambda(\gamma) \geqq \lambda(\gamma')$, so that $\lambda(\gamma') = 0$.

(b) Next we assume that E' is connected, and $E \subset E'$. The sets E' and $bd\,\Omega$ have a positive spherical distance δ, since they are closed and disjoint subsets of Σ. Again, we use the spherical metric, $\rho^*(z)\,|\,dz\,|$, and we consider the family of crosscuts $\{\gamma^0\} = \{\gamma\} - \{\gamma'\}$. If $\{\gamma^0\}$ is empty, $L_{\{\gamma^0\}}(\rho^*) = \infty$ by convention, and we have automatically that $\lambda(\gamma^0) > 0$. If $\{\gamma^0\}$ is not empty, it follows from the connectedness of E' that each γ^0 of this family intersects the set E'; hence, $l_{\rho^*}(\gamma^0) \geqq 2\delta$, and we may conclude as in the proof of the preceding lemma that $\lambda(\gamma^0) > 0$. Since $\{\gamma\} = \{\gamma'\} \cup \{\gamma^0\}$, Lemma 2, together with our assumption that $\lambda(\gamma) = 0$, implies that $\lambda(\gamma') = 0$.

(c) For the general case, where E and E' are arbitrary, we consider a closed and connected subset E^* of Ω which contains both E and E'. (A suitable set E^* can be constructed in view of Lemma 6.) By (b), the sequence $\{a_n\}$ is then a P-sequence with respect to E^*, and hence, by (a), it is also a P-sequence with respect to the set E'. This concludes the proof.

2.3 We now show that every Q-sequence is also a P-sequence.

LEMMA 7. *For every family $\{\tau\}$ of generalized crosscuts of Ω with $\lambda(\tau) = 0$, there is a point $\omega \in bd\,\Omega$ and a sequence $\{\tau_n\}$ of the family, such that the generalized crosscuts τ_n converge to the point ω, in the sense that the sequence $\{r_n\}$, defined by $r_n = \sup\{\psi(z,\omega)\,|\,z \in \tau_n\}$ converges to zero with $1/n$.*

Proof. The spherical metric, $\rho^*(z) = \psi(z,\infty)$, satisfies $0 < A(\rho^*) \leqq \pi$; hence, since $\lambda(\tau) = 0$, $L_{\{\tau\}}(\rho^*) = 0$. Consequently, there is a sequence $\{\tau_n\}$, contained in the family $\{\tau\}$, for which $l_{\rho^*}(\tau_n)$ decreases to 0. We choose a point b_n in each set (τ_n), and we show first that the sequence $\{b_n\}$ cannot have an accumulation point in the interior of the region Ω. For, if ω were such a point, let its spherical distance from the boundary be 3δ. Each τ_n meets the set of points of Ω whose spherical distance from the boundary is less than δ, and an infinite number of the τ_n also meet the spherical neighborhood of ω of radius δ. For these latter curves, we have then that

$l_{\rho^*}(\tau_n) \geqq \delta > 0$, and this contradicts the assumption that $l_{\rho^*}(\tau_n)$ decreases to 0.

We now choose an accumulation point ω of the sequence $\{b_n\}$. ω belongs to the boundary of the region Ω, by the preceding observation. Hence, there is a subsequence, which we again call $\{\tau_n\}$, with the property that the corresponding sequence of points $\{b_n\}$ converges to ω. Consequently, since $l_{\rho^*}(\tau_n)$ converges to zero, we have that $r_n \to 0$.

THEOREM 3. *Let E be a closed subset of the simply connected region Ω. Then the sequence of points $\{a_n\}$ $(a_n \in \Omega)$ is a Q-sequence of Ω with respect to E if and only if it also is a P-sequence of Ω with respect to E.*

Proof. The necessity of the condition is obvious, in view of the Comparison Principle. For the sufficiency, we observe that by Lemma 7 there is a point ω on the boundary of Ω and a sequence $\{\tau_k\}$ contained in the family $\{\tau\}$ which converges to the point ω. Without loss of generality, we may assume that $\omega \neq \infty$. Then the sequence $\{r_k\}$, $r_k = \sup\{|z - \omega| \mid z \in \tau_k\}$ $(k = 1, 2, \cdots)$, converges to zero. Let C_k be the circle with center at ω and with radius r_k $(k = 0, 1, 2, \cdots)$, where r_0 is chosen so small that C_0 separates almost all the crosscuts τ_k from E. For each such k, every simple closed arc, which separates (in the plane) the circles C_0 and C_k, contains a crosscut (indeed a crosscut in the classical sense) which separates (with respect to Ω) E from almost all a_n and which thus belongs to the family $\{\gamma\}$. Consequently, denoting by $\{\delta_k\}$ the family of all such simple closed arcs, we have by Section 1.5 (b) and Lemma 1 that

$$\lambda(\gamma) \leqq \lambda(\delta_k) = 2\pi/\log(r_0/r_k).$$

The right side converges to zero as $k \to \infty$, and the result, $\lambda(\gamma) = 0$, follows.

2.4 DEFINITION 5. *If $\{a_n\}$ and $\{b_m\}$ are two sequences, their join is the sequence $\{c_k\}$, given by $c_{2k-1} = a_k$, and $c_{2k} = b_k$ $(k = 1, 2, \cdots)$.*

DEFINITION 6. *The sequences $\{a_n\}$ and $\{b_m\}$ of points of a simply connected region Ω form a P^*-pair if the join of these sequences is a P-sequence.*

Clearly, each sequence of a P^*-pair is itself a P-sequence, and by Theorem 3, a Q-sequence.

THEOREM 4. *The relation of being a P^*-pair is an equivalence relation on the P-sequences of a region Ω.*

Proof. The reflexivity and symmetry of P^* are obvious; it remains to

show that this relation is also transitive. Let $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ be three P-sequences of Ω, such that the first two and the last two form P^*-pairs. Let E be a closed subset of Ω containing at most a finite number of points of the three sequences. By $\{\tau_{ab}\}$, $\{\tau_{bc}\}$, and $\{\tau_{ac}\}$ we denote the families of generalized crosscuts of Ω which separate E from almost all points of the joins of the sequences indicated by the subscripts. In view of Theorem 3, we know from our hypothesis that

$$\lambda(\tau_{ab}) = \lambda(\tau_{bc}) = 0, \tag{1}$$

and we must show that $\lambda(\tau_{ac}) = 0$.

Now, let τ_1 and τ_2 belong to $\{\tau_{ab}\}$ and $\{\tau_{bc}\}$, respectively. The generalized curve, whose point set is $(\tau_1) \cup (\tau_2)$, separates E from almost all points of the join of $\{a_i\}$ and $\{c_k\}$. Hence, if $(\tau_1) \cup (\tau_2)$ is connected, it forms a member of $\{\tau_{ac}\}$. If, however, that set is not connected, then since each of (τ_1) and (τ_2) is connected, and both separate E from almost all points of $\{b_j\}$, one of (τ_1), (τ_2) must separate the other from E, and it, therefore, belongs to the family $\{\tau_{ac}\}$. Thus, the union of every pair τ_1, τ_2 contains a τ_{ac}. Let $\{\tau^*\}$ be the family of all τ_{ac} with the property that for each τ^* there is $\tau_1 \in \{\tau_{ab}\}$ and $\tau_2 \in \{\tau_{bc}\}$ and such that $\tau^* \subset \tau_1 \cup \tau_2$.

Then we have, for every non-negative function ρ on the region Ω and for every τ^*, that $l_\rho(\tau^*) \leqq l_\rho(\tau_1) + l_\rho(\tau_2)$, where τ_1 and τ_2 are the elements of $\{\tau_{ab}\}$ and $\{\tau_{bc}\}$, respectively, which correspond to τ^*. Hence,

$$L_{\{\tau^*\}}(\rho) \leqq l_\rho(\tau_1) + l_\rho(\tau_2),$$

and finally,

$$L_{\{\tau^*\}}(\rho) \leqq L_{\{\tau_{ab}\}}(\rho) + L_{\{\tau_{bc}\}}(\rho).$$

It is easily seen that ρ is admissible for $\{\tau^*\}$ if and only if it is admissible for both families $\{\tau_{ab}\}$ and $\{\tau_{bc}\}$. We may now restrict ourselves to functions ρ for which $A(\rho) < \infty$, and for these it follows from formula (1) that the right side of the preceding inequality vanishes. Hence, the same is true of the left side, and consequently,

$$\lambda(\tau^*) = \sup \{L^2_{\{\tau^*\}}(\rho)/A(\rho)\} = 0.$$

This, together with the fact that $\{\tau^*\} \subset \{\tau_{ac}\}$, concludes the proof.

2.5 In the case of the unit disk $K = \{z \mid |z| < 1\}$, there is a complete characterization of the P^* equivalence clases.

THEOREM 5. *A sequence $\{a_n\}$ of points of K is a P-sequence if and only if it converges to a point $\zeta = e^{i\theta}$ of the circumference of K.*

Proof.

(a) Let $\lim_{n\to\infty} a_n = \zeta$. We take the set consisting of the origin alone to be E, and set $r_j = |a_j - \zeta|$, and $C_j = \{z \mid |z-\zeta| = r_j\}$. The sequence $\{r_j\}$ converges to 0; without loss of generality we can, by restriction to a suitable subsequence of $\{a_n\}$, assume that C_1 separates C_n (for $n > 1$) from E. Then, if $\{\gamma\}$ is the family of crosscuts of K which separate E from almost all a_n, it follows as in the proof of Theorem 3 that $\lambda(\gamma) = 0$, and thus $\{a_n\}$ is a P-sequence of K.

(b) If $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = e^{i\theta_2}$ are distinct points of the circumference, let $\{a_m\}$ and $\{b_n\}$ be sequences converging to ζ_1 and ζ_2, respectively. Once more, let E be the set consisting of $z = 0$ only. Each member of the family $\{\gamma_{ab}\}$ of crosscuts of K which separate E from almost all points of the join of our two sequences has Euclidean length at least $|\zeta_1 - \zeta_2|$. Denoting by ρ_0 the function $\rho_0(z) \equiv 1$ for all z of K, we have $L_{\{\gamma_{ab}\}}(\rho_0) \geqq |\zeta_1 - \zeta_2|$, while $A(\rho_0) = \pi$; hence,

$$\lambda(\gamma_{ab}) \geqq \tfrac{1}{\pi} |\zeta_1 - \zeta_2|^2 > 0.$$

Consequently, a sequence of points of K with more than one accumulation point on the circumference cannot be a P-sequence.

From the theorem we immediately deduce the following consequence.

COROLLARY. *The P^* equivalence classes of a circular disk are in one-to-one correspondence with the points of its circumference.*

The correspondence in question is the one which assigns to each boundary point the family of all sequences converging to it.

We have remarked repeatedly on the fact that our definitions of P-sequence and P^*-equivalence are formulated in a conformally invariant manner. From this observation, we conclude that if Ω is mapped onto the unit disk K in accordance with the Riemann mapping theorem, then every P-sequence of Ω has an image which converges to a point of the circumference, and two P-sequences have images converging to the same point of the circumfeernce if and only if they form a P^*-pair. Conversely, every sequence of points of K which converges to a point ζ of the circumference is transformed by the inverse map into a P-sequence of the region Ω, and two sequences of K correspond to a P^*-pair if and only if they converge to the same point of the unit circumference.

2.6 We now demonstrate the equivalence of our theory with the classical one. First, we state Carathéodory's main definitions in a suitable

form. Throughout this section, Ω will be a bounded simply connected region; that is the only situation considered by Carathéodory.

DEFINITION 7. *A sequence* $\{\gamma_n\}$ *of crosscuts*[4] *of* Ω *shall be called a chain of crosscuts of* Ω, *if*

(i) *for each* n, γ_{n+1} *separates* γ_n *and* γ_{n+2} *(in* Ω*),*

(ii) *there is a point* $\omega \in bd\,\Omega$ *such that each crosscut* γ_n *of the sequence is an arc of a circle* C_n *with center* ω *and radius* r_n, *and*

(iii) *the sequence* $\{r_n\}$ *decreases to* 0.

The point ω is called the *center* of the chain.

DEFINITION 8. *A sequence* $\{a_n\}$ *of points of* Ω *converges to the prime end determined by the chain of crosscuts* $\{\gamma_n\}$, *provided each crosscut of the chain separates almost all points* a_m *from the component of* $\Omega - \gamma_1$ *not containing* γ_2.

A prime end of Ω is an equivalence class of sequences converging to the same prime end.

A prime end may, of course, be defined by different chains, and these may have distinct centers. If ω can serve as the center of a chain of crosscuts defining a prime end, it is called a *principal point* of the prime end.

Carathéodory has proved the following result, which allows the conclusion that a sequence of points converges to a prime end under conditions on a sequence of crosscuts which do not require that it form a chain.

LEMMA 8. *Let* ω *be a point of the boundary of the bounded simply connected region* Ω, *and let* $\{\gamma_n\}$ *be a sequence of crosscuts which converge to* ω *in the sense that* $\sup_{z \in \gamma_n} |z - \omega| \to 0$. *Furthermore, let* $\{a_n\}$ *be a sequence of points of* Ω *such that each* γ_n *separates almost all* a_m *from a closed set* $E \subset \Omega$. *Then* $\{a_m\}$ *converges to a prime end of* Ω.

For the proof we refer to Carathéodory's paper [4; p. 338].

It is now an easy matter to prove the following theorem:

THEOREM 6. (a) *Each P-sequence* $\{a_n\}$ *of points of* Ω *converges to a prime end of* Ω, *and* (b) *each sequence which converges to a prime end is a P-sequence.*

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[4] For Carathéodory, of course, "crosscut" meant crosscut in the classical sense (see Footnote 2). In view of an obvious modification of the argument of Lemma 7 and Theorem 3, it is not necessary for us to make this distinction here.

Proof. (a) It follows from Lemma 7 that the family of crosscuts $\{\gamma\}$ which separate E from almost all a_n contains a sequence of crosscuts converging to a point $\omega \in bd\,\Omega$. Therefore, Lemma 8 assures that the sequence $\{a_n\}$ converges to a prime end of Ω.

(b) The proof parallels that of Theorem 3. Let $\{\gamma_k\}$ be the chain of circular crosscuts defining the prime end, and let ω be its center. C_k and r_k have the meanings attached to these symbols in Definition 7 (ii). For E we take a closed subset of that component of $\Omega - (\gamma_1)$ which does not contain γ_2. Each simple closed curve, which separates (with respect to the plane) the circles C_1 and C_k, contains a crosscut of Ω separating γ_1 and γ_k (in Ω). This crosscut therefore separates almost all a_n from E. Hence, by Section 1.5 (b) and Lemma 1, we have

$$\lambda(\gamma) \leqq 2\pi/\log\,(r_1/r_k),$$

and the right side converges to 0 as $k \to \infty$, since $\{r_k\}$ is a sequence which decreases to 0. Therefore, $\lambda(\gamma) = 0$, and the proof is complete.

2.7 We now identify our P^*-equivalence classes with the prime ends of Carathéodory. Thus, we replace "$\{a_n\}$ is a P-sequence" by "$\{a_n\}$ converges to a prime end," and "$\{a_n\}$ and $\{b_m\}$ are a P^*-pair" by "$\{a_n\}$ and $\{b_m\}$ converges to the same prime end." Also, if Π is a prime end (i.e., a P^*-equivalence class) of the region Ω and $\{a_n\}$ a sequence of points of Ω, then "$\{a_n\} \in \Pi$" and "$\{a_n\}$ converges to Π" will be considered equivalent.

The following two theorems are Carathéodory's fundamental propositions.

THEOREM 7. *The prime ends of a (bounded) simply connected region are in one-one correspondence with the points of the unit circumference.*

Proof. In view of the conformal invariance of our definitions, this follows from Theorem 5 as we have already indicated in the remarks at the end of Section 2.5.

THEOREM 8. *A sequence of points $\{a_n\}$ of a (bounded) simply connected region Ω either has an accumulation point in Ω, or it contains a subsequence which converges to a prime end of Ω.*

Proof. In the absence of an interior accumulation point, all accumulation point are on the boundary of Ω. We consider the image sequence $\{f(a_n)\}$, where $w = f(z)$ maps Ω conformally and one-one onto the unit disk K of the w-plane. All accumulation points of this image sequence lie on the circumference; we choose a sequence, which converges to a point of

the circumference, and thus, to a prime end of K. The preceding theorem assures that the inverse image of this subsequence converges to a prime end of Ω.

DEFINITION 9. *The set of accumulation points of all P-sequences belonging to the P*-class (the prime end) Π shall be called the point set of Π; we denote it by (Π).*

The point set (Π) of a prime end Π is a closed and connected subset of the boundary; it may reduce to a single point. The principal points of the prime end Π (see Section 2.6) belong to (Π); all other points of (Π) are called *subsidiary points* of Π. The following propositions are stated without proof [4]. The point set of each prime end contains at least one principal point; if it contains more than one, they form a closed connected set; also, there need not be any subsidiary points. In Part III, we shall give a sufficient condition that a P-sequence of points of Π has only principal points of its prime end for accumulation points.

For economy of language it is convenient to identify the prime ends of the unit disk with the points on the circumference to which all P-sequences of the equivalence class converge in view of Theorem 5. Thus, if ζ is a point of the circumference, we shall freely employ terminology like "the prime end ζ."

The correspondence of Theorem 7 is used to define the cyclic order of the prime ends of a simply connected region Ω. A segment of prime ends is a collection of prime ends corresponding to a segment of the circumference. The notion of cyclic order could also be introduced intrinsically, i.e. using only concepts and constructions relating to the region Ω. (See, for example, Kerékjártó's book ([5], p. 80), or the author's thesis.)

III. Central Convergence.

3.0 Summary. In this final section we study the structure of prime ends in more detail. Namely, at the end of Section 2.7, we distinguished between two types of points, principal and subsidiary, on the point set of a prime end. It is now our concern to place into a conformally invariant setting a theorem of Lindelöf which gives a sufficient condition that all accumulation points of a P-sequence be principal.

3.1 DEFINITION 10. *If all accumulation points of a P-sequence of points of Ω are principal points of the prime end to which the sequence converges, we call the sequence a principal P-sequence.*

Being a principal sequence in this sense is *not* a conformally invariant property. However, we give a sufficient condition that a P-sequence be principal; this condition, which adapts Lindelöf's theorem on approach in an angle, is a conformally invariant one. Indeed, with each point a_n of the P-sequence under consideration we associate an extremal length λ_n, and we formulate our theorem in terms of a certain boundedness condition on the sequence $\{\lambda_n\}$.

Suppose then that Π and Π^* are two prime ends of Ω which correspond, when Ω is mapped conformally and one-to-one onto K, to $+i$ and $-i$, respectively. Let α be the segment of prime ends corresponding, under the inverse of this map, to the (open) arc $(+i, -1, -i)$ of the circumference, and let α^* be the segment of prime ends corresponding to the open arc $(-i, +1, +i)$. Again indicating point sets of prime ends by parentheses, we have

$$bd\,\Omega = (\Pi) \cup (\alpha) \cup (\Pi^*) \cup (\alpha^*).$$

Of course, the preceding is not, in general, a union of disjoint sets.

We shall say that a crosscut of the region Ω terminates on the segment α if every P-sequence of points of Ω which lies on the crosscut belongs to a prime end of α.

With a point $a \in \Omega$ and the pair of prime ends Π and Π^* of Ω we associate two families of crosscuts of Ω, $\{\gamma\}$ and $\{\gamma^*\}$. Namely, each $\gamma \in \{\gamma\}$ shall separate the point a from α^* and terminate on α; the family $\{\gamma^*\}$ is defined dually. From Theorem 1, we have that

$$\lambda(\gamma) \cdot \lambda(\gamma^*) = 4. \tag{1}$$

This formula also follows from an explicit computation of $\lambda(\gamma)$; in fact,

$$\lambda(\gamma) = 4\nu(\cos \tfrac{1}{2}\pi\omega_\alpha(a)), \tag{2}$$

where $\omega_\alpha(a)$ is the harmonic measure with respect to Ω of the segment α at a, $\nu(r) = \frac{1}{2}K^*(r)/K(r)$,

$$K(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-\frac{1}{2}}\,dx,$$

and $K^*(r) = K((1-r^2)^{\frac{1}{2}})$.

3.2 DEFINITION 11. *If $\{a_j\}$ is a P-sequence belonging to the prime end Π, then it is a central P-sequence, provided the associated sequence of positive numbers, $\{\lambda(\gamma_j) + 1/\lambda(\gamma_j)\}$ is bounded.*

It is a consequence of formula (2), above, that this definition is independent of the auxiliary prime end Π^*.

In view of formula (1) of Section 3.1, $\{a_j\}$ is a central P-sequence if and only if neither of the sequences $\{\lambda(\gamma_j)\}$ nor $\{\lambda(\gamma^*_j)\}$ has a subsequence which converges to either 0 or ∞.

The theorem of Lindelöf [6; p. 28] reads, in terms of central convergence, as follows.

THEOREM 9. *If $\{a_j\}$ is a central P-sequence of points of Ω, then $\{a_j\}$ is a principal P-sequence.*

Proof. We suppose that an accumulation point of the P-sequence $\{a_j\}$, say ω, is a subsidiary point of the prime end Π to which the sequences converges; we show that no subsequence, which converges to ω, can be a central P-sequence. It follows that the original sequence was not central.

Let the sequence $\{a_j\}$ converge to the subsidiary point ω; for convenience we assume $\omega \neq \infty$. We choose a prime end Π^* of Ω distinct from Π and a P-sequence $\{b_k\}$ which belongs to Π^*. Now let C be a circle with center at the point ω and radius r. Let the positive number r be so small that (a) almost all a_j lie inside C, (b) almost all b_k lie outside C, and (c) none of the arcs of $\Omega \cap (C)$ separates the set E (employed in the definition of the prime ends Π and Π^*) from almost all a_j. This choice of r is always possible, since $a_j \to \omega$, $\Pi^* \neq \Pi$, and ω is a subsidiary point of Π.

If σ_j is a crosscut of Ω which lies on C and which separates a_j from E, it has both its endpoints on the segment α, or it has both endpoints on α^*. The collection $\{\sigma_j\}$ is not finite, because of (a) and (c). Therefore, there must actually be an infinite number of distinct crosscuts σ_j whose endpoints are either both on α or on α^*.

Without loss of generality, we now assume that an infinite number of the σ_j have endpoints on α^*; we restrict ourselves to those values of the subscript j for which this condition holds. By Ω_j we denote the component of $\Omega - (\sigma_j)$ which contains a_j. Then every prime end of Ω_j that does not belong to σ_j is also a prime end of the region Ω, and it belongs to the segment α^*. Consequently, every crosscut γ of the family $\{\gamma_j\}$ must contain a crosscut of Ω_j both of whose endpoints lie on σ_j, and which intersects the circle C_j with center at ω and radius $r_j = |a_j - \omega|$.

Hence, every crosscut of $\{\gamma_j\}$ contains a crosscut of the family connecting C and C_j. From Lemma 1 and Section 1.5 (b), we now deduce that

$$\lambda(\gamma_j) \geqq \tfrac{1}{2\pi} \log (r/r_j),$$

so that $\lambda(\gamma_j) \to \infty$, since $r_j \to 0$. This shows that $\{a_j\}$ is not a central P-sequence. The proof is complete.

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REFERENCES.

[1] L. V. Ahlfors, *Conformal Mapping.* Lecture notes (transcribed by R. Osserman), Oklahoma A. and M. College, 1951.

[2] L. V. Ahlfors and A. Beurling, "Conformal Invariants," in *Construction and Applications of Conformal Maps* (*Proceedings of a Symposium*), Washington, D. C., National Bureau of Standards, Applied Mathematics Series, No. 18, 1952.

[3] L. V. Ahlfors and A. Beurling, "Conformal invariants and function-theoretic null-sets," *Acta Mathematica*, vol. 83 (1950), pp. 101-129.

[4] C. Carathéodory, "Über die Begrenzung einfach zusammenhangender Gebiete," *Mathematische Annalen*, vol. 73 (1913), pp. 323-370.

[5] B. v. Kerékjártó, *Vorlesungen über Topologie*, vol. 1, Berlin, J. Springer, 1923.

[6] E. Lindelöf, "Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme," *Acta Societatis Scientiarum Fennicae*, vol. 46 (1915), No. 4, pp. 1-35.

[7] S. Mazurkiewicz, "Über die Definition der Primenden," *Fundamenta Mathematicae*, vol. 26 (1936), pp. 272-279.

[8] H. Renggli, "Zur konformen Abbildung auf Normalgebiete," *Commentarii Mathematici Helvetici*, vol. 31 (1956), pp. 5-40.

[9] H. D. Ursell and L. C. Young, "Remarks on the theory of prime ends," *Memoirs of the American Mathematical Society*, No. 3 (1951).

ALGEBRAIC GROUPS AND THE GALOIS THEORY OF DIFFERENTIAL FIELDS.*

By ELLIS KOLCHIN[1] and SERGE LANG.

The purpose of this note is to draw together morely closely the Galois theory of differential fields and the theory of algebraic groups and homogeneous spaces.

Given a strongly normal extension of a differential field of characteristic 0, it is known (see [2] or Matsumura [3]) that the Galois group is isomorphic to the group of points of an algebraic group which are rational over the field of constants. We show below that the extension itself has a model which is a principal homogeneous space for the algebraic group, such that the operation of the elements of the Galois group on the extension is induced by the operation of the corresponding elements of the algebraic group on the space. The Galois theory can then be deduced from known facts concerning principal homogeneous spaces (Rosenlicht [4]). This provides an analysis of the Galois theory into two parts, one depending on the foundations of differential algebra and the other on the foundations of the theory of algebraic groups.

There remains almost untouched a converse problem which, for the sake of simplicity, we state for connected algebraic groups, or as we shall also say, group varieties. Given a differential field F of characteristic 0 with algebraically closed field of constants C, a group variety G defined over C, and a principal homogeneous space V for G defined over F, to determine all extensions of the differential field structure on F to the function field $F(V)$ which makes $F(V)$ a strongly normal extension of F with automorphism group induced by the group G_C of points of G rational over C. Put another way, the problem is to describe all possible ways of extending the derivation operators of F to derivation operators on $F(V)$ which are invariant relative to G_C, which commute with each other, and which have precisely C as field of constants. We observe here merely that the first condition alone

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presents no difficulty, i.e. that it is easily shown that the derivations of F can be extended to invariant derivations of $F(V)$.

1. Algebraic groups. We recall here briefly the notions of algebraic group and principal homogeneous space. For the general theory, see Weil [5] and [6], and Rosenlicht [4].

An *algebraic set* V is a finite union of varieties, $V = \bigcup V_i$, which for the purposes of this paper we shall always assume disjoints; the V_i are the components of V. Let V, W be two algebraic sets. A rational map $f: V \to W$ is an algebraic set, having one component f_i for each component V_i of V such that f_i is a rational map in the usual sense of V_i into a component of W. We say that f is defined at a point $u \in U_i$ if f_i is defined at that point, and we write $f(u) = f_i(u)$.

An *algebraic group* is an algebraic set G together with an everywhere defined rational map of $G \times G$ into G, relative to which G is a group, having the further property that the inverse mapping $x \to x^{-1}$ is an everywhere defined rational map of G into G.

If G is an algebraic group, a *transformation space* for G is an algebraic set V together with an everywhere defined rational map of $V \times G$ into V (the operation of G on V) which, if we denote the image of (v, x) by vx, satisfies the conditions

$$v(xy) = (vx)y \text{ and } ve = v$$

for all $v \in V$, $x \in G$, $y \in G$ (e denoting the unity element of G.) A transformation space is a *homogeneous space* if, given any $v, w \in V$, there exists an $x \in G$ such that $vx = w$. The homogeneous space is *principal* if this point x is determined uniquely and rationally. By the latter we mean that there exists an everywhere defined rational map $\lambda: V \times V \to G$ such that $x = \lambda(v, w)$. If V is a principal homogeneous space for G then the number of components of V equals that of G and each component of V is a principal homogeneous space for the component G_0 of G containing e, G_0 being a normal subgroup of G the cosets of which are precisely the components of G.

We shall say that the algebraic group G is defined over a field k if all the components of G, of its group law, and of its inverse mappings are defined over k. Similarly, we shall say that the principal homogeneous space V for G is defined over k if G is defined over k, and if all the components of V, of the operation of G on V, and of λ are also defined over k. Although this notion of field of definition may present some drawbacks, it is adequate for our present purpose.

2. Differential-algebraic preliminaries. Let F be a differential field of characteristic 0. Denote the field of constants of F by C. We work in a fixed universal extension F^* of F in the sense of [1], and denote the field of constants of F^* by C^*. The facts stated in the following propositions were proved in [1].

PROPOSITION 1. *The fields F and C^* are linearly disjoint over C.*

By an extension of F, we shall always mean a differential field extension contained in the universal domain. Similarly, an isomorphism will always mean a differential field isomorphism. When we wish to consider only the field structure, we shall say field extension, and field isomorphism.

Let E be an extension of F, and suppose that as a field extension E is finitely generated. Let F_0 be the algebraic closure of F in E. We recall if σ and σ' are isomorphisms of E, σ' is called a *specialization* of σ if the family $(\sigma'\alpha)_{\alpha\in E}$ is a specialization of $(\sigma\alpha)_{\alpha\in E}$ over E in the usual sense of algebraic geometry. The specialization is called *generic* if also σ is a specialization of σ'. An isomorphism of E over F is called *isolated* if it is not a nongeneric specialization of any isomorphism of E over F.

PROPOSITION 2. (a) *There exists a finite set of isolated isomorphisms of E over F such that every isomorphism of E over F is a specialization of precisely one of these.*

(b) *An isomorphism σ of E over F is isolated if and only if the transcendence degree of the compositum $E\cdot\sigma E$ over E equals that of E over F.*

(c) *An element of E invariant under an isolated isomorphism of E over F belongs to F_0.*

(d) *If σ is isolated then σ' is a specialization of σ if and only if their restriction to F_0 coincide. Every isomorphism of F_0 over F is the restriction to F_0 of an isolated isomorphism of E over F.*

Suppose further that E has the same field of constants C as F has, and that C is algebraically closed. We recall that an isomorphism σ of E is called *strong* if $\sigma E \subset E\cdot C^*$ and $E \subset \sigma E\cdot C^*$. Letting $C(\sigma)$ denote the field of constants of $E\cdot\sigma E$, we can see with the help of Proposition 1 that σ is strong if and only if

$$E\cdot C(\sigma) = E\cdot\sigma E = \sigma E\cdot C(\sigma).$$

Every automorphism of E is obviously a strong isomorphism of E.

PROPOSITION 3. (a) *Every strong isomorphism of E over F has a specialization which is an automorphism of E.*

(b) *An isomorphism of E over F is strong if and only if it is the restriction to E of an automorphism of $E \cdot C^*$ over $F \cdot C^*$.*

As the automorphism mentioned in (b) is evidently unique, we may identify the strong isomorphism with it. The set of strong isomorphisms of E over F is then identified with the group of automorphisms of $E \cdot C^*$ over $F \cdot C^*$. We denote this group by $\mathfrak{G}$.

Let $\sigma, \tau \in \mathfrak{G}$. As $C(\sigma^{-1})$ is the field of constants of $E \cdot \sigma^{-1}E$, $\sigma C(\sigma^{-1}) = C(\sigma^{-1})$ must be the field of constants of $\sigma(E \cdot \sigma^{-1}E) = E \cdot \sigma E$. Thus

$$C(\sigma^{-1}) = C(\sigma).$$

Writing $C(\sigma, \tau)$ to denote the compositum $C(\sigma) \cdot C(\tau)$, we see that

$$E \cdot C(\sigma, \sigma\tau) = E \cdot C(\sigma) \cdot C(\sigma\tau) = E \cdot \sigma E \cdot \sigma\tau E = E \cdot \sigma(E \cdot \tau E) = E \cdot \sigma(E \cdot C(\tau))$$
$$= E \cdot \sigma E \cdot C(\tau) = E \cdot C(\sigma) \cdot C(\tau) = E \cdot C(\sigma, \tau).$$

It follows with the help of Proposition 1 (applied to E instead of F) that $C(\sigma, \tau) = C(\sigma, \sigma\tau)$. Therefore $C(\tau^{-1}, \sigma^{-1}) = C(\tau^{-1}, \tau^{-1}\sigma^{-1}) = C(\tau, \sigma\tau)$. Thus

$$C(\sigma, \sigma\tau) = C(\sigma, \tau) = C(\sigma\tau, \tau). \tag{1}$$

We observe also, using Propositions 2(b) and Proposition 1, that $\sigma \in \mathfrak{G}$ is isolated if and only if the transcendence degree of $C(\sigma)$ over C equals that of E over F.

We suppose, finally, that the extension E of F is *strongly normal*, that is, that in addition to all the hypotheses made above, every isomorphism of E over F is strong. $\mathfrak{G}$ is then called the *Galois group* of E over F.

3. The theorem and its proof. We temporarily impose the restriction that $F = F_0$. Then by Proposition 2(a) and (d), all isolated elements of $\mathfrak{G}$ are generic specializations of each other.

Let $\sigma_0 \in \mathfrak{G}$ be isolated, and let W be any model of the field $C(\sigma_0)$ over C. This means that W is a variety defined over C and there exists a generic point x of W over C such that $C(\sigma_0) = C(x)$. If σ is any isolated element of $\mathfrak{G}$ then σ is a generic specialization of σ_0 so that there exists an isomorphism $E \cdot \sigma_0 E \approx E \cdot \sigma E$ over E mapping $\sigma_0\alpha$ onto $\sigma\alpha$ $(\alpha \in E)$. This isomorphism maps $C(\sigma_0)$ onto $C(\sigma)$ and therefore maps x onto a generic point of W over C, which we call x_σ (so that in particular $x_{\sigma_0} = x$) and which has the property that $C(\sigma) = C(x_\sigma)$. Because $E \cdot \sigma_0 E = E \cdot C(\sigma_0)$ the iso-

morphism $E \cdot \sigma_0 E \approx E \cdot \sigma E$ over E is determined by its restriction to $C(\sigma_0)$, so that σ is determined by x_σ.

Furthermore if y is any point of W which is rational over C^* and generic over C, there exists a unique field isomorphism $C(x) \approx C(y)$ over C mapping x onto y, and because E and C^* are linearly disjoint over C this field isomorphism can be extended to a field isomorphism ϕ over E of $E \cdot C(x) = E \cdot C(\sigma_0) = E \cdot \sigma_0 E$ onto $E \cdot C(y)$. As ϕ maps $C(\sigma)$ into $C(y)$, ϕ commutes on $C(\sigma)$ with the derivation operators of the differential field structure, and as ϕ is the identity on E, ϕ commutes on E with these derivation operators, so that ϕ commutes on $E \cdot C(\sigma)$ with these operators. In other words, ϕ is a differential field isomorphism. Letting $\tau = \phi\sigma_0$, we see that $y = x_\tau$. We have thus proved that $\sigma \to x_\sigma$ is a bijective mapping of the set of isolated elements of G onto the set of points of W which are rational over C^* and generic over C.

Let $\sigma, \tau \in \mathfrak{G}$. If in addition to being isolated, σ and τ are independent, in the sense that $C(\sigma)$ and $C(\tau)$ are linearly disjoint over C, then (1) shows that $\sigma\tau$ is isolated. Furthermore, by Proposition 1, $E \cdot \sigma E = E \cdot C(\sigma)$ and $E \cdot \tau E = E \cdot C(\tau)$ are linearly disjoint over E. If σ' and τ' are also independent isolated elements of $\mathfrak{G}$, then the isomorphism $E \cdot \sigma E \approx E \cdot \sigma' E$ over E mapping $(\sigma\alpha)_{\alpha \in E}$ onto $(\sigma'\alpha)_{\alpha \in E}$ and the analogous isomorphism $E \cdot \tau E \approx E \cdot \tau' E$ have a unique common extension.

$$E \cdot \sigma E \cdot \tau E \approx E \cdot \sigma' E \cdot \tau' E. \tag{2}$$

For any element $\alpha \in E$, $\tau\alpha$ can be expressed in the form $\tau\alpha = f(x_\tau)$, where f is a rational function on W defined over E. Hence we have $\tau'\alpha = f(x_{\tau'})$, $\sigma\tau\alpha = f^\sigma(x_\tau)$, and $\sigma'\tau'\alpha = f^{\sigma'}(x_{\tau'})$. From this it is clear that the isomorphism (2) maps $(\sigma\tau\alpha)_{\alpha \in E}$ onto $(\sigma'\tau'\alpha)_{\alpha \in E}$ and therefore maps $x_{\sigma\tau}$ onto $x_{\sigma'\tau'}$. Thus $(x_{\sigma'}, x_{\tau'}, x_{\sigma'\tau'})$ is a specialization of $(x_\sigma, x_\tau, x_{\sigma\tau})$ over C (and even over E).

It follows that we may define a law of composition on W by the formula $x_\sigma x_\tau = x_{\sigma\tau}$, this law of composition being defined over C. From (1) we have $C(x_\sigma, x_\sigma x_\tau) = C(x_\sigma, x_\tau) = C(x_\sigma x_\tau, x_\tau)$. Furthermore, if ω is an isolated element of $\mathfrak{G}$ such that ω, σ, τ are independent, i.e. such that $x_\omega, x_\sigma, x_\tau$ are independent generic points of W, then it is clear that $x_\omega(x_\sigma x_\tau) = (x_\omega x_\sigma)x_\tau$. Thus we have a normal law of composition on W, so that by the fundamental theorem of Wiel [5], W is birationally equivalent over C to a group variety defined over C. Thus we may suppose that W is itself a group variety, and we henceforth denote it by G.

Since we have assumed that $F = F_0$, E is a regular extension of F.

Therefore there exists a model V for E over F, i.e. a variety V defined over F such that E may be written $E = F(v)$ where v is a generic point of V over F. Letting σ be an isolated element of $\mathfrak{G}$, so that x_σ is a point of G which is rational over C^* and generic over C (and therefore over E), we define a rational map of $V \times G$ into V by the formula

$$(v, x_\sigma) \to vx_\sigma = \sigma v.$$

It follows from the above considerations that this rational map, which is defined over F, does not depend on which isolated element σ is used. It is also easy to see from what precedes that if σ and τ are independent isolated elements of $\mathfrak{G}$, then

$$F(vx_\sigma, x_\sigma) = F(v, x_\sigma) = F(v, vx_\sigma)$$

and

$$v(x_\sigma x_\tau) = (vx_\sigma)x_\tau.$$

It follows from another result of Weil ([6] Prop. 3) that V is birationally equivalent over F to a principal homogeneous space for G defined over F, and therefore we may assume that V itself is such a space. This principal homogeneous space is, moreover, uniquely determined up to an everywhere defined birational correspondence compatible with the operation of G.

This being the case, we see that every point $a \in G$ (generic or not) such that v is generic on V over $F(a)$ induces a field isomorphism σ_a of $E = F(v)$ over F mapping v onto va. If a is any point of the group G_{C^*} of points of G rational over C^*, then v is automatically generic on V over $F(a)$, because E and C^* are linearly disjoint over C, and hence E and $F \cdot C^*$ are linearly disjoint over F. In this case, σ_a is an isomorphism of the differential field structure. Indeed, for elements of G_{C^*} which are generic over C this has been proved above, and for an arbitrary $a \in G_{C^*}$, there exists an element $x \in G_{C^*}$ such that x^{-1} and xa are generic over C, and $\sigma_a = \sigma_{x^{-1}}\sigma_{xa}$.

The mapping $a \to \sigma_a$ $(a \in G_{C^*})$ is obviously an isomorphism of G_{C^*} into $\mathfrak{G}$. It is actually onto $\mathfrak{G}$, for given any $\tau \in \mathfrak{G}$ we can select an isolated $\tau' \in \mathfrak{G}$ so that τ and τ' are independent. Then $\tau\tau'$ and τ'^{-1} are isolated and $\tau = (\tau\tau')\tau'^{-1}$.

We have now proved our main result in the special case that $F = F_0$. We shall indicate below how to extend it to the general case, and we state here the general theorem which for the case $F = F_0$ summarizes the above discussion.

THEOREM. *Let F be a differential field of characteristic 0 with algebraically closed field of constants C, and let E be a strongly normal extension*

of F, all lying in a universal extension F^* with field of constants C^*. Let $\mathfrak{G}$ be the Galois group of E over F. Then there exists an algebraic group G defined over C, and a principal homogeneous spave V for G, defined over the algebraic closure F_0 of F in E, having the following properties.

(a) *V is a model of E over F, that is, any component of V has a point v generic over F_0 such that $E = F_0(v)$, and these components are all conjugates to each other over F.*

(b) *For each $\sigma \in \mathfrak{G}$ there exists a unique point $x_\sigma \in G_{C^*}$ (the group of points of G which are rational over C^*) such that $\sigma v = v x_\sigma$, and the mapping $\sigma \to x_\sigma$ $(\sigma \in \mathfrak{G})$ is a group isomorphism of $\mathfrak{G}$ onto G_{C^*}.*

(c) *This isomorphism is rational in the sense that $C(\sigma) = C(x_\sigma)$ and that σ' is a specialization of σ if and only if $x_{\sigma'}$ is a specialization of x_σ over C.*

In order to extend the result proved for $F = F_0$ to the general case, we apply the proved result to the extension E of F_0. We thus obtain a connected algebraic group G_0 defined over C and a principal homogeneous space V_0 defined over F_0 such that V_0 has a point v generic over F_0 with $E = F_0(v)$, and the Galois group $\mathfrak{G}_0$ of E over F_0 is isomorphic with G_{0C^*} in the manner described in the statement of the theorem. Every $\sigma \in \mathfrak{G}$ maps F_0 onto the set of elements of σE which are algebraic over F, so that if σ is an automorphism of E then $\sigma F_0 = F_0$. Since by Proposition 3, σ always has a specialization which is an automorphism of E, which by Proposition 2(d) coincides with σ on F_0, we see that $\sigma F = F_0$ for every $\sigma \in \mathfrak{G}$. It now follows by Proposition 2(d) and (a) that the mapping

$$\sigma \to \text{restriction of } \sigma \text{ to } F_0$$

is a homomorphism of $\mathfrak{G}$ with kernel $\mathfrak{G}_0$ onto a finite group of automorphism of F_0 with fixed field F. If σ_i $(i = 1, \cdots, r)$ are the elements of the finite set of itsolated isomorphisms mentioned in Proposition 2(a) (with say $\sigma_0 \in \mathfrak{G}_0$) and if $\mathfrak{G}_i$ is the set of specializations of σ_i, then the $\mathfrak{G}_i$ are cosets of $\mathfrak{G}_0$ in $\mathfrak{G}$.

Using the fact (Proposition 3(a)) that each $\mathfrak{G}_i$ contains an element σ_i' which is an automorphism of E, we follow [2] to extend the isomorphism $\mathfrak{G}_0 \approx G_0$ to an isomorphism $\mathfrak{G} \approx G$ (rational in the sense of (c)), where G is an algebraic group of which the component of the unity element is G_0. The varieties $V_i = V_0^{\sigma_i'}$ determined by the restrictions of the σ_i' to F_0 are precisely the conjugates of V_0 over F, and for each i the point $v_i = \sigma_i' v_0$ is a generic point of V_i over F_0. We now define the operation of G on the

algebraic set $V = \bigcup V_i$ in the obvious manner, namely by setting $v_i x_{\sigma_j} = \sigma_j v_i$ for all i and j. We leave to the reader the verification that this makes V into a principal homogeneous space for G with the stated properties.

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REFERENCES.

[1] E. R. Kolchin, "Galois theory of differential fields," *American Journal of Mathematics*, vol. 75 (1953), pp. 753-824.

[2] ———, "On the Galois theory of differential fields," *American Journal of Mathematics*, vol. 77 (1955), pp. 868-894.

[3] H. Matsumura, "Automorphism-groups of differential fields and group varieties," *Memoirs of the College of Science, University of Kyoto, Series A*, vol. 28 (1954), pp. 283-292.

[4] M. Rosenlicht, "Some basic theorems on algebraic groups," *American Journal of Mathematics*, vol. 78 (1956), pp. 401-443.

[5] A. Weil, "On algebraic groups of transformation," *American Journal of Mathematics*, vol. 77 (1955), pp. 355-391.

[6] ———, "On algebraic groups and homogeneous spaces," *American Journal of Mathematics*, vol. 77 (1955), pp. 493-512.

STRATIFICATIONS OF CAUCHY'S "STABLE" TRANSCENDENTS AND OF MITTAG-LEFFLER'S ENTIRE FUNCTIONS.*

By AUREL WINTNER.

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1. Let $\alpha > 0$ and, if x is real and $0 \leqq q < 1$, put

$$\theta_\alpha(x, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^\alpha} \cos nx.$$

What will be proved below, when combined with a known procedure based on Poisson's summation formula (cf. [13]), leads to the following result: Whenever $\beta > \alpha$, there exists on the interval $0 \leqq r < 1$ a *monotone* bounded function $\lambda(r) = \lambda_{\alpha\beta^q}(r)$ in such a way that

$$\theta_\alpha(x, q) = \int_0^{1-0} \theta_\beta(x, r)\, d\lambda_{\alpha\beta^q}(r) \qquad (d\lambda = d_r\lambda \geqq 0).$$

Since $\theta_2(x, q)$ is an elliptic theta-function which, in view of its Eulerian factorization, is positive throughout, there follows, as a corollary ($\beta = 2$), the known fact that $\theta_\alpha(x, q) > 0$ holds for every real x and for every positive $q < 1$ whenever $\alpha \leqq 2$. The integral formula becomes curious also under the stricter assumption $\alpha < 1$, since $\theta_\beta(x, q)$ is Poisson's explicit function $(1 - 2q\cos x + q^2)^{-1}$ if $\beta = 1$. But the point in the general "stratification" rule of $\theta_\alpha(x, q)$ is that arbitrarily large values of β, and any positive α which is less than β, are allowed for every positive $q < 1$. Of particular interest are the cases $\beta = 4, 6, \cdots$, occurring in the Hardy-Littlewood treatment of Waring's problem.

2. It was shown by Cauchy ([2], pp. 99-101) that the densities of probability $F(x)$, $-\infty < x < \infty$, of what today are called symmetric stable distributions must be the Fourier transforms $F(x) = F_\alpha(x)$ of the functions $\exp(-|t|^\alpha)$, where $\alpha > 0$; so that, if $x \geqq 0$ and $t \geqq 0$ without loss of generality, then

$$F_\alpha(x) = \frac{1}{\pi} \int_0^{\infty} \exp(-t^\alpha) \cos xt\, dt \tag{1}$$

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* Received October 8, 1957.

(the Fourier transform of (1) is absolutely convergent, since $F_\alpha(x) = O(x^{-a})$, where $a = 1 + \alpha > 0$, and even $x^a F(x) \to c$, where $c = c(\alpha)$, holds as $x \to \infty$, with $c(\alpha) = 0$ if and only if $\alpha = 2, 4, 6, \cdots$; cf. [11], p. 187). Cauchy's result on (1) the "stability" of (1) was rediscovered by Edgeworth [4] (Pizzetti's analysis of the questions involved mentions only Edgeworth; cf. [9], p. 114).

Cauchy and his immediate successors tacitly assumed that every function (1) actually is a density function, which would mean that $F_\alpha(x) \geqq 0$ holds for all x whenever $\alpha > 0$ (and Cauchy found for $\alpha > 2$ the curious "paradox," forgotten in the meantime, which is quoted by Czuber [3], p. 184). But it was observed by F. Bernstein [1] that this cannot be the case if $n = 4$ (and his method, adapted from Hardy's proof for the existence of an infinity of odd zeros on the critical line of the Riemann zeta-function, applies to $n = 6, 8, \cdots$ also).

The final result is due to P. Lévy (cf., e.g., [6], chap. VI of the second part, or [8], chap. V), who proved that $F_\alpha(x) \geqq 0$ holds for every x if and only if $\alpha \leqq 2$. But while the necessity of $\alpha \leqq 2$ can be proved easily, most proofs for the sufficiency of $\alpha \leqq 2$ are variants of one and the same existence consideration.

In what follows, the existence statement will be modified so as to contain a result which holds for *every* $\alpha > 0$ (i.e., not only for $\alpha \leqq 2$) and to contain, nevertheless, Lévy's result ($F_\alpha \geqq 0$) if $\alpha \leqq 2$ (even though the latter result cannot be true if $\alpha > 2$). In fact, it turns out that, *whenever* $0 < \alpha < \beta < \infty$, *there exists on the half-line* $0 \leqq u < \infty$ *a monotone function* $\phi(u) = \phi_{\alpha\beta}(u)$ *satisfying* $\phi(0) = 0$, $\phi(\infty) = 1$ *and*

$$F_\alpha(x) = \int_0^\infty F_\beta(xu)\, d\phi_{\alpha\beta}(u) \qquad (d\phi_{\alpha\beta}(u) \geqq 0) \tag{2}$$

$(0 \leqq x < \infty)$. Since (1) reduces to $C \exp(-\frac{1}{2}x^2) > 0$ if $\alpha = 2$, the case $\beta = 2$ of (2) proves that not only $F_\alpha(x) \geqq 0$ but even

$$F_\alpha(x) > 0 \text{ for all } x \text{ if } \alpha \leqq 2. \tag{3}$$

But the point is that (2) holds for every β (even for $\beta > 2$; e.g., for $\beta = 4, 6, \cdots$) whenever $\alpha < \beta$ (concerning the case of $F_\beta(x)$ when β is an even integer exceeding 2, cf. Pólya's note [11] and his asymptotic formula which I quoted on p. 834 of vol. 78 (1956) of this Journal).

It will also follow that, whenever $0 < \alpha < \beta < \infty$, the distribution $\phi_{\alpha\beta}(u)$ occurring in (2) is absolutely continuous on the closed half-line $[0, \infty)$ and

regular (in the sense of analyticity) at every point u of the open half-line $(0,\infty)$.

3. For the point of departure in the proof of the first italicized theorem, it is possible to choose (among various, substantially equivalent choices; cf. e.g., the second paragraph of § 5 below) the existence of those "stable distributions" which may be called *unilateral.* These distributions, which in contrast to the *symmetric* stable distributions (which exist for $a \leqq 2$; cf. § 2), exist only for $a < 1$, were introduced by Lévy (cf. [6], pp. 260-262, or [8], pp. 178-179). After a normalization of the unit of length on the x-axis, the unilateral stable distributions, which will be denoted by $\psi(x) = \psi_a(x)$, where $-\infty < x < \infty$ and $0 < a < 1$, are constant either for $-\infty < x \leqq 0$ or for $0 \leqq x < \infty$, and their Fourier-Stieltjes transforms

$$\int_{-\infty}^{\infty} e^{iux} d\psi(x), \text{ where } d\psi(x) \geqq 0 \text{ and } \int_{-\infty}^{\infty} d\psi(x) = 1, \tag{4}$$

are identical either with

$$\exp(-|u|^a[-1 + i\tan(\tfrac{1}{2}a\pi)\operatorname{sgn} u]), \text{ where } -\infty < u < \infty, \tag{5}$$

or with the complex conjugate of (5).

Since both (4) and (5) go over into their complex conjugates if u is replaced by $-u$, it is sufficient to consider the case in which (4) is identical with (5) itself, where $\psi = \psi_a$. Then $d\psi_a(x) = 0$ for $-\infty < u \leqq 0$; hence

$$\exp(-u^a + iu^a \tan \tfrac{1}{2}a\pi) = \int_0^{\infty} e^{iux} d\psi_a(x), \tag{6}$$

if, as always from now on, *u is confined to the half-line* $0 \leqq u < \infty$ (this involves no loss, since (4) goes over into its complex conjugate if u is replaced by $-u$).

What will be needed of all of this can be summarized as follows: If $0 < a < 1$, then there exists on the x-half-line $[0,\infty)$ a *monotone* function ψ_a (with $\psi(0) = 0$, $\psi(\infty) = 1$) satisfying (6) on the u-half-line $[0,\infty)$.

4. Change the unit of length on the x-axis in such a way that u becomes replaced by cu, where $c = c_a$ is the positive constant $(\cos \frac{1}{2}\pi a)^{1/a}$, and denote the function of x which thus results from $\psi_a(x)$ again by $\psi_a(x)$. Then $\psi_a(0) = 0$, $\psi_a(\infty) = 1$ and $d\psi_a(x) \geqq 0$ hold again, but (6) becomes simply

$$\exp(-iu)^a = \int_0^{\infty} e^{iux} d\psi_a(x), \tag{7}$$

since the parenthetical expression on the left of (6) becomes replaced by $-a_\alpha u^\alpha + ib_\alpha u^\alpha = (e^{-\frac{1}{2}\pi i}u)^\alpha$, where

$$(8) \qquad a_\alpha = \cos \tfrac{1}{2}\pi\alpha, \qquad b_\alpha = \sin \tfrac{1}{2}\pi\alpha \qquad (0 < \alpha < 1).$$

For every α (< 1), the analytic behavior of $\psi_\alpha(x)$ on $[0,\infty)$ is as follows: $\psi_\alpha(x)$ is (i) of class C^∞ on $[0,\infty)$ and (ii) analytic (regular) at every point $x > 0$ of $[0,\infty)$ but (iii) singular at the end-point, $x = 0$, of $[0,\infty)$, since all derivatives of $\psi_\alpha(x)$ vanish at $x = 0$ (so that the analytic behavior of $\psi_\alpha(x)$ is the same as that of the function which is $\exp(-x^{-1})$ for $x > 0$ and 0 for $x \leqq 0$).

In fact, if a_α and b_α are defined by (8), then $\psi_\alpha'(x) = d\psi_\alpha(x)/dx$ can be represented as the following explicit Laplace transform:

$$(9) \qquad \psi_\alpha'(x) = \tfrac{1}{\pi}\int_0^\infty e^{-xt}\exp(-a_\alpha t^\alpha)\sin(b_\alpha t^\alpha)\,dt, \text{ where } 0 < x < \infty;$$

cf. [14], p. 128 (where the parameter β is 0 in the present *unilateral* case). Clearly, (ii) is implied by (9). On the other hand, (i) and (iii) can be obtained, after some explicit labor, by letting $x \to +0$ in (9). But it is easier to proceed as follows:

If $\psi_\alpha(x)$ is defined to be 0 for $-\infty < x \leqq 0$, then, on the one hand, the integral (7) becomes (4) and, on the other hand, (i) and (iii) together become equivalent to the statement that $\psi_\alpha(x)$ is of class C^∞ on $(-\infty,\infty)$. But Fourier's inversion formula shows that $\psi(x)$ is certainly of class C^∞ on $(-\infty,\infty)$ if, for every fixed n, the function (4) of u is $O(|u|^{-n})$ as $|u| \to \infty$. Finally, the case (7) of (4) satisfies this O-condition, since the absolute value of the function on the left of (7) is $\exp(-|u|^\alpha)$, where $\alpha > 0$.

5. Since $\psi_\alpha(x)$ is monotone and bounded on the x-half-line $[0,\infty)$, and since (7) is valid on the open u-half-line $(0,\infty)$ (incidentally, at $u = +0$ as well), it is clear, by absolutely-uniform convergence and for reasons of analyticity, that (7) remains valid if $-iu$ is replaced by $-z$, where z is any point of the half-plane $\operatorname{Re} z > 0$. Hence, if z is chosen to be real, it follows that

$$(10) \qquad \exp(-x^\alpha) = \int_0^\infty e^{-xu}\,d\psi_\alpha(u), \text{ where } d\psi_\alpha(u) \geqq 0, \qquad (0 < \alpha < 1),$$

holds on the open x-half-line $(0,\infty)$ and, for trivial reasons of continuity, on the closure $[0,\infty)$ of $(0,\infty)$ as well.

In view of the Hausdorff-S. Bernstein theorem, the existence of a function ψ_α satisfying (10) on $[0,\infty)$ does not state more than that the n-th derivative of $\exp(-x^\alpha)$ is non-negative or non-positive on $[0,\infty)$ according as n is even or odd; and, since $0<\alpha<1$ by assumption, this follows by a trivial induction from n to $n+1$. Conversely, since the steps in the preceding deduction (6) $\Rightarrow$ (7) $\Rightarrow$ (10) can of course be retraced, an application of the existence theorem of Hausdorff-Bernstein, when applied to $\exp(-x^\alpha)$, supplies the existence of Lévy's unilateral stable distributions.*

In any case, (9) supplies the explicit form of the distribution occurring in (10). Actually, (9) will not be used in what follows (although the corollary (ii) of (9) in § 4 will be needed in order to obtain the additional information claimed in the last paragraph of § 2).

6. In order to prove the first italicized assertion of § 2, let α, β be any pair of indices satisfying $0<\alpha<\beta<\infty$ and let t vary on the half-line $[0,\infty)$. Then (10) is applicable when α and x are replaced by α/β and t respectively. Hence

$$\exp(-t^\gamma)=\int_0^\infty e^{-tu}d\psi_\gamma(u), \text{ where } \gamma=\alpha/\beta. \tag{11}$$

But $[0,\infty)$ goes over into itself if t is replaced by t^c, where c is any positive constant. Choose $c=\beta$. Then, since $\psi_\gamma(0)=0$, $\psi_\gamma(\infty)=1$ and $d\psi_\gamma(u)\geqq 0$, it follows from (11), by placing $v=u^{1/\beta}$, that there exists on the v-half-line $[0,\infty)$ a monotone function $\psi(v)=\psi_{\alpha\beta}(v)$ satisfying $\psi(0)=0$, $\psi(\infty)=1$ and

$$\exp(-t^\alpha)=\int_0^\infty \exp\{-(tv)^\beta\}\,d\psi_{\alpha\beta}(v), \tag{12}$$

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* It is clear that the formal fact used above can be phrased more generally, as follows: If $d\mu(u)$ is a *symmetric* distribution on $(-\infty,\infty)$ and $F(t)=F(-t)$ denotes its characteristic functon (i.e., the integral of $\exp(ixu)d\mu(u)$ or $\cos(xu)d\mu(u)$ over $(-\infty,\infty)$ for a fixed real t, where $d\mu(u)+d\mu(-u)=0$, $d\mu(u)\geqq 0$ and $F(0)=1$), and if $F(t)$ is totally monotone for positive t, then $F(i\mid t\mid \operatorname{sgn} t)$ is the characteristic function of a certain *unilateral* distribution $d\lambda(u)\geqq 0$ on $(-\infty,\infty)$, with $d\lambda(u)=0$ on $(-\infty,0)$, and vice versa. For, by the Hausdorff-Bernstein theorem, the total monotony of $F(t)$ means the existence of a distribution $d\lambda(u)$ on $[0,\infty)$ for which the integral of $\exp(-tu)d\lambda(u)$ over $[0,\infty)$ becomes $F(t)$ at every $t>0$.

The symmetry of the original distribution $d\mu(u)$ on $(-\infty,\infty)$ is implied, of course, by the circumstance that its characteristic function $F(t)$ is real-valued (hence, even). The introduction of Hausdorff-Bernstein transforms into the theory of probability is due to P. Lévy; cf., e.g., his note in the *Comptes Rendus*, vol. 207 (1938), pp. 1368-1370, where these transforms occur in connection with certain (more general than stable) distribution laws of indefinite divisibility.

where $0 \leqq t < \infty$ and

$$\int_0^\infty |\, d\psi_{\alpha\beta}(v)\,| = \int_0^\infty d\psi_{\alpha\beta}(v) = 1 < \infty. \tag{13}$$

Let $x \geqq 0$, multiply (12) by $\cos(xt)$, and integrate the product over the half-line $0 \leqq t < \infty$ (while x is fixed). It is seen from the definition (1) that what then results differs from the assertion (2) only in notation, provided that it is legitimate to interchange the order of integrations with respect to t and v. Finally, it is clear from (13) and from the parenthetical remark made after (1) that the legitimacy of this interchange is assured by Fubini's theorem.

The proof of the stratification theorem (2) of Cauchy's transcendents (1) is now complete. The explicit form of the derivative of the (absolutely continuous) function $\phi_{\alpha\beta}(u)$ occurring in (2) follows from (9) and from the substitutions which lead from (11) and (12) to (12) and (2) respectively.

7. Let finally be mentioned a simple proof of the fact that $F_\alpha(x) \geqq 0$ cannot hold for all $x > 0$ if $\alpha > 2$.

If α is arbitrary (> 0), it follows from the parenthetical remark made after (1) that, if F_α is denoted by F, the integral

$$\int_0^\infty F(x) \cos tx \, dx \tag{14}$$

is absolutely convergent (at $t = 0$ and, therefore, at every t). On the other hand, (14) is the Fourier inversion of (1) and is therefore identical with $\exp(-t^\alpha)$, where $0 \leqq t < \infty$. But the latter function is $1 + O(t^\alpha)$, as $t \to 0$, for every $\alpha > 0$; hence, it is $1 + o(t^2)$ if $\alpha > 2$.

Consequently, it is sufficient to ascertain that if $F(x)$ is any function which is absolutely integrable on the half-line $0 \leqq x < \infty$ and which has the property that the corresponding function (14) is $1 + o(t^2)$ as $t \to 0$, then, unless $F(x)$ vanishes identically, $F(x) \geqq 0$ cannot hold for all x. But this can be seen as follows (cf., e.g., [12], p. 33):

Since (15) is a continuous function of t and is $1 + o(t^2)$ as $t \to 0$, it follows, by placing $t = 0$ in (14) and subtracting the result from (14) itself, that

$$\int_0^R F(x) \sin^2(\tfrac{1}{2}tx)\, dx = o(t^2), \text{ where } t \to 0.$$

holds for $R=\infty$ and therefore, if $F(x)\geqq 0$ throughout, for any positive function $R=R(t)<\infty$. Since $\sin^2 u$ exceeds a positive constant multiple of u^2 if $0\leqq u\leqq \frac{1}{2}$, it follows, by choosing $R(t)=1/t$, that, as $t\to 0$,

$$\int_0^{1/t} F(x)\,(tx)^2 dx = o(t^2), \text{ hence } \int_0^{1/t} F(x)x^2\,dx = o(1),$$

and so $\int_0^{\infty} F(x)x^2 dx = 0$. Since $F(x)\geqq 0$, this proves the identical vanishing of F (for almost all x).

8. The proof of (2) depended on the formal steps (6) $\Rightarrow$ (7) $\Rightarrow$ (10) $\Rightarrow$ (2) and rested, therefore, on the following fact (cf. the second paragraph of § 5): If $f_\alpha(x)=\exp(-x^\alpha)$, where $0\leqq x<\infty$, then there exists for $0\leqq u<\infty$ one monotone function $\phi_{\alpha\beta}(u)$ for which

$$(*) \qquad f_\alpha(x)=\int_0^{\infty} f_\beta(xu)\,d\phi_{\alpha\beta}(u), \text{ where } d\phi_{\alpha\beta}(u)\geqq 0,$$

is an identity in x, where α and β are arbitrary parameters satisfying $0<\alpha<\beta<\infty$. Instances of such a stratification rule (*) which are more on the surface than is its instance $f_\alpha(x)=\exp(-x^\alpha)$ are $f_\alpha=g_\alpha$ and $f_\alpha=h_\alpha$, where $g_\alpha(x)$ denotes $(1+x)^{-\alpha}$, while $h_\alpha(x)$ denotes $(1-x)^{\alpha-1}$ or 0 according as $0\leqq x<1$ or $1\leqq x<\infty$.

First, if $x>0$ and $\beta>\alpha>0$, then the integral of $(x+u)^{-\beta}u^{-1+\beta-\alpha}$ over the u-half-line $[0,\infty)$ is convergent and (as seen by placing $u=xv$ at a fixed x) has a value which is independent of x (the value of this positive constant $C=C(\alpha,\beta)$ follows from one of the definitions of the beta-function). Hence, if x and u are replaced by $x+1$ and $u-1$ respectively, it follows that

$$(1+x)^{-\alpha}=c_{\alpha\beta}\int_1^{\infty} (u+x)^{-\beta}(u-1)^{-1+\beta}du, \text{ where } c_{\alpha\beta}>0,$$

holds on the open x-half-line $(-1,\infty)$ and so, in particular, on $[0,\infty)$. The satisfaction of (*) on $[0,\infty)$ by $f_\alpha=g_\alpha$, where $g_\alpha(x)=(1+x)^{-\alpha}$, now follows if u is replaced by $1/u$ in the last integral (so that $d\phi_{\alpha\beta}(u)=0$ in (*) if $1\leqq u<\infty$).

Clearly, the satisfaction of (*) by $f_\alpha=h_\alpha$ can be verified in the same way.

9. It is clear that if (*) is satisfied by a family $f_\alpha(x)$, then it is also

satisfied (I) by the family $f_\alpha(x^\mu)$, where μ is any positive constant, and (II) by the family

$$\int_0^\infty k(xs) f_\alpha(s)\, ds,$$

where $k(t)$ is an "arbitrary" function of $t(\geqq 0)$, subject (relative to f_α) to the applicability of Fubini's theorem.

For instance, (2) resulted by choosing $f_\alpha(x) = \exp(-x^\alpha)$ and $k(t) = \cos t$ in (II). But $k(t) = \sin t$ (or, more generally, $k(t) = t^\gamma Z(t)$, where $Z(t)$ is a cylindrical function, such as $J_\lambda(t)$ or $K_\lambda(t)$, and γ an appropriate index, leading to the applicability of Fubini's theorem) could also have been allowed. Thus (2) remains true if $\cos xt$ is replaced by $\sin xt$ in (1).

In order to illustrate (II) *and* (I), apply first (I) alone, by choosing in it $\mu = 2$ and either $f_\alpha = g_\alpha$ or $f_\alpha = h_\alpha$, where g_α, h_α are the functions of § 8. It then follows that (*) holds for $f_\alpha(x) = (1+x^2)^{-\alpha-\frac{1}{2}}$, where $0 \leqq x < \infty$, as well as for the function $f_\alpha(x)$ which is $(1-x^2)^{\alpha-\frac{1}{2}}$ or 0 according as $0 \leqq x < 1$ or $1 \leqq x < \infty$ (in both cases, $0 < \alpha < \infty$). Hence, if (II) is applied to $k(t) = \cos t$ in both cases, the "real integral" definition of the cylindrical functions $J_\lambda(t)$, $K_\lambda(t)$ shows that (*) holds, if $\beta > \alpha > 0$, for $f_\alpha(x) = J_{\alpha-\frac{1}{2}}(x)/x^{\alpha-\frac{1}{2}}$ and $f_\alpha(x) = K_{\alpha-\frac{1}{2}}(x)/x^{\alpha-\frac{1}{2}}$ and, therefore (with some $\phi_{\alpha\beta}$), for $f_\alpha = J_{\alpha-\frac{1}{2}}$ and $f_\alpha = K_{\alpha-\frac{1}{2}}$ and so, in particular, for $f_\alpha = J_\alpha$ and $f_\alpha = K_\alpha$.

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10. Corresponding to the stratification theorem (2) of Cauchy's transcendents (1), where α (>0) is arbitrary, there is (in terms of some family of distribution ϕ) a stratification theorem

$$E_\alpha(z) = \int_0^\infty E_\beta(zu)\, d\phi_{\alpha\beta}(u), \qquad (d\phi_{\alpha\beta}(u) \geqq 0), \tag{15}$$

where $E_\alpha(z)$ denotes Mittag-Leffler's entire function

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1+\alpha n) \tag{16}$$

and z can be complex in (15): *If* $0 < \alpha < \beta < \infty$, *then there exists on the half-line* $0 \leqq u < \infty$ *a monotone function* $\phi(u) = \phi_{\alpha\beta}(u)$ *satisfying* $\phi(0) = 0$, $\phi(\infty) = 1$ *and* (15).

The (absolute) convergence of the integral (15) at every point z of the

complex plane is part of the assertion. The proof can be based on the following fact, verified in [15] along the lines of pp. 61-64 of my paper in vol. 58 (1936) of this Journal: Corresponding to every index θ on the range $0<\theta<1$, there exists on the line $-\infty<t<\infty$ a monotone function $\psi(t)=\psi_\theta(t)$ in terms of which the identity

$$\Gamma(z+1)/\Gamma(\theta z+1)=\int_{-\infty}^{\infty} e^{-zt}d\psi_\theta(t) \qquad (d\psi_\theta(t)\geqq 0) \tag{17}$$

holds on the half-plane $\operatorname{Re} z>-1$ and so, in particular, on the half-line $z\geqq 0$.

Choose $z=\beta x$, where $\beta>0$ and $x\geqq 0$, put $\alpha=\theta\beta$, and replace t by $-\log s$ in (17). Then, if $-\psi(-\log s)$ is denoted by $\phi(s)$, where $0<s<\infty$, it is seen that, whenever $0<\alpha<\beta$ (i.e., $0<\theta<1$ for $\theta=\alpha/\beta$), the relation

$$\Gamma(\beta x+1)/\Gamma(\alpha x+1)=\int_0^{\infty} s^x\,d\phi(s), \text{ where } 0\leqq x<\infty,$$

holds for a certain monotone function $\phi(s)=\phi_{\alpha\beta}(s)$ satisfying $\phi(0)=\phi(+0)=0$ and $\phi(\infty)=1$ (in fact, the gamma-quotient on the left becomes 1 at $x=0$). If x and s are now replaced by n and u respectively, it follows that *the Stieltjes moment problem*

$$\int_0^{\infty} u^n\,d\phi_{\alpha\beta}(u)=\Gamma(\beta n+1)/\Gamma(\alpha n+1) \qquad (n=0,1,\cdots) \tag{18}$$

has a solution $d\phi_{\alpha\beta}(u)\geqq 0$ *whenever* $\beta>\alpha>0$.

Since all coefficients of (16) are positive, it is sufficient to prove (15) for positive z. But then a term-by-term integration (at *this* stage, possibly with the result $\infty=\infty$) is allowed in what results if (16), with α and z (>0) replaced by ψ and zu respectively, is inserted on the right of (15). Hence (15) is equivalent to

$$E_\alpha(z)=\sum_{n=0}^{\infty} z^n\int_0^{\infty} u^n d\phi_{\alpha\beta}(u). \tag{19}$$

Since the truth of (19) is clear from (18) and (16), the proof is complete.

The simplest particular case of the general stratification theorem of (16), a particular case to which (16) cannot be reduced directly, is a result of W. Feller (found by him by entirely different means, namely, by an appeal

to Lévy's result on the case $\alpha < 2$ of (1); cf. [10]), which states that $E_\gamma(-x)$ is totally monotone on $[0,\infty)$ if $0 < \gamma < 1$ (and, of course, if $\gamma = 1$). This fact relates to (15) in about the same way as (10) relates to (2). In order to see this, replace α, β and z in (15) by γ, 1 $(> \gamma > 0)$ and $-x$ respectively, and denote the case $\alpha = \gamma$, $\beta = 1$ of $\phi_{\alpha\beta}$ by ϕ_γ. Then (15) goes over into

$$E_\gamma(-x) = \int_0^\infty e^{-xu} d\phi_\gamma(u), \qquad (0 < \gamma < 1), \tag{20}$$

since $E_1(z) = e^z$, by (16). Since $d\phi_\gamma(u) \geqq 0$, and since (20) holds for every x, it follow from (20) that $E_\gamma(-x)$ is totally monotone on $(-\infty,\infty)$ or, equivalently, on $[0,\infty)$. If x is replaced by $-x$ in (16), this can be formulated as follows: *If* $0 < \gamma < 1$, *then* $E_\gamma(x)$ *and all of its derivatives are positive for* $-\infty < x < \infty$ (this is clear from (16) only for $0 \leqq x < \infty$ but then, of course, for every $\gamma = \alpha > 0$).

Remark. There results a stratification theorem corresponding to (2) and (16) if Cauchy's and Mittag-Leffler's transcendents, (1) and (16), are replaced by the entire function

$$e_\alpha(z) = \sum_{n=0}^\infty z^n/\Gamma(\alpha + n), \tag{21}$$

considered (in a slightly different form) by Euler and (precisely in this form) in Heaviside's formal work on fractional differentiation (for $0 < \alpha < 1$ and, more particularly, for $\alpha = \frac{1}{2}$).

Both (16) and (21) reduce to e^z if $\alpha = 1$. But it so happens that *the proviso* $0 < \alpha < \beta$ *of* (15) *must be changed to* $0 < \beta < \alpha$ *if* (16) *is replaced by* (21). In other words, corresponding to every pair of indices satisfying $0 < \alpha < \beta$, there exists on the half-line $0 \leqq u < \infty$ a function $\phi(u) = \phi_{\alpha\beta}(u)$ for which

$$e_\beta(z) = \int_0^\infty e_\alpha(zu)\, d\phi_{\alpha\beta}(u), \text{ where } \phi_{\alpha\beta}(0) = 0,\ \phi_{\alpha\beta}(\infty) = 1 \text{ and } d\phi_{\alpha\beta}(u) \geqq 0, \tag{22}$$

holds as an identity in z.

This counterpart of (15) lies quite on the surface. In fact, instead of (18), only the definition,

$$\int_0^1 u^{s-1}(1-u)^{t-1}\, du = \Gamma(s)\Gamma(t)/\Gamma(s+t), \qquad (s > 0, t > 0), \tag{23}$$

of the beta-function is now needed. In addition, $d\phi_{\alpha\beta}(u) = 0$ for $1 < u < \infty$ in (22), and the existence statement of (22) can be replaced by the explicit formula

$$\Gamma(\beta - \alpha)e_\beta(z) = \int_0^1 e_\alpha(zu)u^{\alpha-1}(1-u)^{\beta-\alpha-1}du, \tag{24}$$

where $0 < \alpha < \beta < \infty$. For, if the definition (21) is inserted on the right as well as on the left of (24), comparison of like powers of z shows that (24) is equivalent to what results if (23) is applied as follows: $s = \alpha + n$ and $t = \beta - \alpha$, where $n = 0, 1, \cdots$ (so that, since $0 < \alpha < \beta$, the parenthetical proviso of (23) is satisfied for every $n \geqq 0$).

APPENDIX.

On a theorem of F. Bernstein.

Let s be a positive constant (to play the part of a standard deviation or of its reciprocal value) and put $\epsilon(x) = \exp(-x^2)$. Then, since (1) reduces to a constant multiple of an $\epsilon(sx)$ if $\alpha = 2$, it is clear from the case $\beta = 2$ of (2) that Lévy's theorem, asserting the positivity of $F_\alpha(x)$ when $\alpha < 2$, is equivalent to the statement that, if $G(t) = G_s(t)$ denotes, for a fixed $x > 0$, the "Gauss-Fourier transform"

$$G(t) = \int_0^\infty \cos(tx)\epsilon(sx)g(x)\,dx, \text{ where } \epsilon(y) = \exp(-y^2), \tag{1}$$

of a function $g(x)$, where $0 \leqq x < \infty$, then

$$G(t) > 0, \text{ where } 0 \leqq t < \infty \tag{2}$$

(or, equivalently, $-\infty < t < \infty$), holds for a certain $g = g_\alpha$ (if $0 < \alpha < 2$).

Quite another class of functions g for which (1) satisfies (2) (for every $s > 0$) was considered by F. Bernstein [1] (cf. also Mathias' subsequent paper [2]; in particular, his Satz 13). Starting with a certain expansion of an "arbitrary" $g(x)$, subject to restrictions of integrability (and, for some reason, of analyticity) of the functions occurring in the expansion, he considers a "Hankel kernel" $K(x, y)$, a kernel which is a function $K(x+y)$ of $x+y$ alone; and, in terms of certain properties (in this connection, cf. [3], pp. 328-329) of a bilinear expansion of $K(x+y) = g(x+y)$, he specifies a condition under which (1) will satisfy (2). Finally, by an applica-

tion of the addition theorem of elliptic functions to $K(x+y)=\mathrm{cn}(x+y)$, Bernstein proves that his conditions are fulfilled, hence (2) holds for (1), if

$$(3) \qquad g(x)=\mathrm{cn}\,x, \text{ where } \mathrm{cn}\,x=\mathrm{cn}(x,k) \text{ and } {}^{*}\ 0\leqq k<1$$

(this Jacobian elliptic function $g(x)=g_k(x)$ is real-valued, even and periodic, with a period $T=T(k)$ which tends to 2π as $k\to 0$ and to ∞ as $k\to 1$).

It does not seem to have been observed in the literature that this result of Bernstein has actually little to do either with his bilinear expansions or with such a specific matter as the addition theorem of elliptic functions. In fact, it so happens that (2) must be satisfied by (1) whenever (i) the function $g(x)$ is continuous, even, periodic (of an arbitrary period T) and such that $a_n\geqq 0$ holds for every n, and $a_n>0$ for some n, in its Fourier series

$$(4) \qquad g(x)\sim\sum_{n=0}^{\infty}a_n\cos(2\pi nx/T)=g(x), \text{ where } 0=\sum_{n=0}^{\infty}a_n=\sum_{n=0}^{\infty}|\,a_n\,|<\infty;$$

more generally, whenever (ii) there exists on the half-line $0\leqq\lambda<\infty$ a monotone function $f(\lambda)$ satisfying

$$(5) \quad g(x)=\int_0^{\infty}\cos(\lambda x)\,df(\lambda), \text{ where } 0<f(\infty)-f(0)=\int_0^{\infty}|\,df(\lambda)\,|<\infty.$$

Bernstein's result on (3) follows from (i), since the classical Fourier series (Jacobi; cf. e.g., [3], p. 330) of (3) puts into evidence the truth of $a_n\geqq 0$ in the case (3) of (4). [In this connection, it is worth mentioning that also the results of Stieltjes ([4], pp. 549-555; cf [3], pp. 329-330) on the Laplace transform of (3) and a corresponding Stieltjes transform depend merely on the satisfaction of the assumption $a_n\geqq 0$ of (4) by (3).]

Curiously enough, the trivial limiting case $k=0$ of Bernstein's result on (3), the case in which (3) and the assertion (2) for (1) reduce to $g(x)=\cos x$ and

$$(6) \qquad \int_0^{\infty}\cos(tx)\exp(-sx^2)\cos x\,dx>0 \qquad (s>0)$$

respectively, is sufficient in order to prove not only the generalization (i) of Bernstein's result but even the generalization (ii) of (i).

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* Actually, the limiting case $k=1$ need not be excluded. This follows either for reasons of continuity or, more directly, by observing that, on the one hand, $\mathrm{cn}(x,k)=1/\cosh x$ if $k=1$, and, on the other hand, the value of $G(x)$ is known to be given by an explicit cosh-formula, rendering $G(x)>0$ trivial, if $g(x)=1/\cosh x$.

In fact, it is clear from reasons of distributivity that, in order to prove (ii), it is sufficient to verify the satisfaction of (*2*) by (*1*) for the case in which the function $f(\lambda)$ occurring in (*5*) is a step-function consisting of a single saltus (>0, which can be assumed to be 1). But (*2*) then becomes equivalent to what results when the second cosine is replaced by $\cos(\lambda_0 x)$ in (*6*), where $f(\lambda) = \frac{1}{2}\operatorname{sgn}(\lambda - \lambda_0)$. After a change of the unit of length on the x-axis in (*6*), this can be reduced to the two particular cases $\lambda_0 = 0$, $\lambda_0 = 1$. If $\lambda_0 = 0$, then, since $\exp(-\frac{1}{2}t^2)$ is its own Fourier (cosine) reciprocal, $G(t)$ becomes $a\exp(-bt^2)$, where $a > 0$ and $b \geqq 0$ are constants; so that (*2*) is true if $\lambda_0 = 0$. If $\lambda_0 = 1$, then $G(t)$ is the integral (*6*) (as it stands now). Finally, the truth of the inequality (*6*) need not be concluded from the case $k = 0$ of Bernstein's result on (*3*). In fact, an application of the limiting case ($\cos(\alpha \pm \beta) = \cdots$; $k = 0$) of the addition theorem of the elliptic function (*3*) shows that the integral (*6*) can be written as the sum of two functions of the form $a\exp(-bt^2)$, where $a > 0$, $b \geqq 0$.

What all of this actually verifies is nothing but the fact that, under the assumption (*5*) of (iii), the function (*1*), instead of satisfying just (*2*), is a totally monotone function of $t^{\frac{1}{2}}$, where $0 \leqq t < \infty$. In particular, $G(t)$ is a decreasing function (unless $G(t) = \text{Const.}$, i. e., unless $f(\lambda) = \text{const. sgn}\,\lambda$), a fact not revealed by Bernstein's proof (*loc. cit.*) of (*2*) even in his case (*3*).

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REFERENCES.

[1] F. Bernstein, "Über das Fourierintegral . . .," *Mathematische Annalen*, vol. 79 (1920), pp. 265-268.

[2] A. Cauchy, *Œuvres complètes*, ser. 1, vol. 12 (1900 [1853]).

[3] E. Czuber, "Die Entwicklung der Wahrscheinlichkeitstheorie und ihrer Anwendungen," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 7 (1898), pp. 1-271.

[4] F. Y. Edgeworth, "The law of error," *Philosophical Magazine*, ser. 5, vol. 16 (1883), pp. 300-309.

[5] P. Lévy, "Sur les lois stables en calcul des probabilités," *Comptes Rendus*, vol. 176 (1923), pp. 365-366.

[6] ———, *Calcul des probabilités*, Paris, 1925.

[7] ———, *Théorie de l'addition des variables aléatoires*, Paris, 1937.

[8] ———, *Processus stochastiques et mouvement brownien*, Paris, 1948.

[9] P. Pizzetti, *I fondamenti matematici per la critica dei risultati sperimentali*, Genova, 1891.

[10] H. Pollard, "The completely monotonic character of the Mittag-Leffler function $E_a(-x)$," *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 1115-1116.

[11] G. Pólya, "On the zeros of an integral function represented by Fourier's integral," *Messenger of Mathematics*, vol. 52 (1923), pp. 185-188.

[12] A. Wintner, *Lectures on Asymptotic Distributions and Infinite Convolutions*, Institute for Advanced Study (Planographed by Edwards Brothers, Inc.), 1938.

[13] ———, "On the shape of the angular case of Cauchy's distribution curves," *Annals of Mathematical Statistics*, vol. 18 (1947), pp. 589-593.

[14] ———, "Stable distributions and Laplace transforms," *Annali della Scuola Normale Superiore di Pisa*, ser. 3, vol. 10 (1956), pp. 127-134.

[15] ———, "On Heaviside's and Mittag-Leffler's generalizations of the exponential function, the symmetric stable distributions of Cauchy-Lévy, and a property of the gamma-function," to appear in the Fréchet jubilee volume of the *Journal de Mathématique* (1958).

APPENDIX.

[1] F. Bernstein, "Ein Kriterium für den positiv definiten Charakter von Fourierintegralen und die Darstellung solcher als Summe von Quadraten," *Mathematische Annalen*, vol. 85 (1922), pp. 155-159.

[2] M. Mathias, "Über positive Fourier-Integrale," *Mathematische Zeitschrift*, vol. 16 (1923), pp. 103-125.

[3] O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig-Berlin, 1913.

[4] T. J. Stieltjes, *Œuvres complètes*, vol. 2, Groningen, 1918.

ON COSINE-LIKE ARCHES.*

By AUREL WINTNER.

Let $f(x)$ be a function possessing derivatives of arbitrarily high order on an open x-interval (a,b), and let $(-1)^n D^{2n} f(x) \geqq 0$, where $D = d/dx$, hold on (a,b) for $n = 0, 1, 2, \cdots$. If the trivial case $f(x) = \text{const.}\ (\geqq 0)$ is excluded, then, for reasons of convexity, this infinity of inequalities cannot be satisfied on an unbounded interval (a,b), and so it can be assumed that $(a,b) = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The resulting class of functions on the interval

$$-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi \tag{1}$$

will be called *cosine-like*, since $f(x) = \cos x$ is the simplest (non-constant) function satisfying

$$(-1)^n D^{2n} f(x) \geqq 0 \text{ on } (1) \text{ for } n = 0, 1, \cdots. \tag{2}$$

In fact, (2) means that the graph of each of the functions $f, -D^2 f, D^4 f, -D^6 f, \cdots$ is an arc (over $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$) which is situated in the closure of the upper half-plane (in fact, within the open upper half-plane unless $f(x) = \text{const.} = 0$) and which is (strictly) convex from above.

Correspondingly, a straightforward adaptation of a classical argument (initiated, for the class of continuous convex functions, by Jensen and further developed, for the class of totally monotone functions, by S. Bernstein) shows that, if difference, instead of differential, quotients are used in the definition of the class of cosine-like functions, then the assumption of indefinite differentiability, or even the existence of a first derivative, need not be introduced explicitly.

The general methods of Bernstein also supply, as a corollary, the following information: If $f(x)$ is cosine-like, then it can be continued from the given interval (1) into the z-plane and leads to an entire function $f(z)$ which, in addition, must be of exponential type. Only the first of these two results, proved also by Widder, will be used in what follows. References will be found in the paper of Boas and Pólya, Duke Math. Journ., vol. 9 (1942), pp. 406-407; for an exposition of the relevant results of Bernstein and his successors, cf. Chap. 11 of Boas' *Entire Functions*, New York, 1954.

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A more delicate (as a matter of fact, a *structural*) function-theoretical information on what is necessary for a cosine-like function is contained in the following theorem:

If a function $f(x)$, given on (1), *satisfies* (2), *then there exists on the closed t-interval* $[-1, 1]$ *a non-decreasing function* $\phi(t)$ *having the following properties: If* $p(w)$ *denotes the function which is defined by*

$$p(w) = \sum_{n=0}^{\infty} \mu_n w^n, \text{ where } \mu_n = \int_{-1}^{1} t^n d\phi(t) \tag{3}$$

(hence, $|\mu_n| \leqq \mu_0$*), on the circle*

$$|w| < 1 \tag{4}$$

or, equivalently, by

$$p(w) = \int_{-1}^{1} (1 - tw)^{-1} d\phi(t) \qquad (d\phi(t) \geqq 0) \tag{5}$$

on the simply-connected domain S which results if both half-lines

$$-\infty < w \leqq -1, \ 1 \leqq w < \infty \tag{6}$$

are removed from the w-plane, and if the single-valued, regular function $q(w)$ *is defined on* (4) *or, equivalently, on S, by*

$$q(w) = (1 - w^2)^{\frac{1}{2}} p(w), \text{ where } q(0) \geqq 0, \tag{7}$$

then $q(\sin z)$ *is an entire function (of exponential type) of z satisfying*

$$f(x) = q(\sin x) \text{ on } (1). \tag{8}$$

The functions $f(x)$ satisfying (8), with (7) and (5) for some $\phi(t)$, and rendering $q(\sin z)$ an entire function,† are not quite so scarce as it might seem from the function-theoretical restrictions needed for (7) and (5), and from Stieltjes' inversion formula of his transform (5) of $d\phi$ (cf., e.g.,

† If this restriction, which is essential indeed, is disregarded, then there results a *formal* connection between (7)-(8) and a relation occurring in the theory of orthogonal polynomials. In fact, non-negative functions of the type

$$p(u) = q(u)/\sqrt{(1 - u^2)}, \qquad (-1 < u < 1, \text{ or } u = \sin x),$$

where however $p(u)$ need not satisfy anything like the regularity conditions imposed on it by (5) and the further restriction specified before (8), play there a considerable part, as densities of absolutely continuous weights (cf. Chap. XI-XIII of Szegö's *Orthogonal Polynomials*, New York, 1939; the reason for the formal connection is clear from § 11.5).

§ 43-§ 45 of my *Spektraltheorie*, Leipzig, 1929). For instance, since (2) is satisfied if $f(x) = \cos(vx)$, where v is any value on the interval $0 \leqq v \leqq 1$, it is satified if

$$f(x) = \int_0^1 \cos(vx)\, d\lambda(v), \tag{9}$$

where $\lambda(v)$ is any non-decreasing function. Nor does (2) imply that the entire function $f(z)$ is even (as (9) is). For, since (2) is satisfied by $f(x) = \cos(vx)$ if $0 < v < 1$, it is satisfied by $f(x) = \cos(vx - \beta)$ if $\beta = \beta(v)$ is small enough, and the latter $f(x)$ can be "stratified" (with regard to v) by means of a weight $d\lambda(v) \geqq 0$, as in (9) (and this time for $-1 \leqq v \leqq 1$). But by the theorem to be proved, all these functions $f(x)$ on (1) must be such as to possess a representation (8) in which the corresponding $d\phi(t)$, occurring in (5), satisfies the restrictions implicit in the condition that *the product of* $\cos z$ *and*

$$p(\sin z) = \int_{-1}^1 (1 - t \sin z)^{-1}\, d\phi(t) \qquad (d\phi(t) \geqq 0) \tag{10}$$

becomes an entire function of z. In this regard, it is instructive to choose $\phi(t) = \frac{1}{2} \operatorname{sgn}(t - t_0)$, where t_0 is any point of the interval $[-1, 1]$. Then $d\phi(t) \geqq 0$ is satisfied. But (10) and (7) show that $f(z) = q(\sin z)$ now becomes the meromorphic function

$$f(z) = W/(1 - t_0 W), \text{ where } W = \sin z \text{ and } -1 \leqq t_0 \leqq 1, \tag{11}$$

a function of z which fails to be an entire function (and, correspondingly, $f(x)$ fails to be cosine-like) unless $t_0 = 0$, hence $f(z) = W = \cos z$. On the other hand, if $\phi(t) = t$, then (10) leads to logarithmic branch-points, instead of poles.*

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* If $\mathrm{Cn}(x, k)$ denotes what results from Gudermann's $\mathrm{cn}(t, k)$ (i. e., Jacobi's $\cos \mathrm{am}\, t$), if t is replaced by $2Kx/\pi$, then $y(x) = \mathrm{Cn}(x, k)$, where k is any fixed positive modulus < 1, is the most immediate generalization of $\cos x = \mathrm{Cn}(x, 0)$. But the function $y = y(x) = y_k(x)$, where $y_k = \mathrm{Cn}(x, k)$, though always even, of period 2π, positive on (1) and decreasing on the upper half of (1), must fail to be cosine-like for *every* positive k, since, because of its double periodicity for *complex* x, it has *poles*, except in the limiting case $k = 0$ of $y = \cos x$ itself.

Incidentally, even the case $n = 1$ of (1), instead of being satisfied by $f(x) = \mathrm{Cn}(x, k)$ for every positive $k (< 1)$, is satisfied only for $0 \leqq k \leqq k_0$, where k_0 is an absolute constant which is less than 1. In fact, $y = \mathrm{Cn}(x, k)$ is a *convex* arch over (1) if and only if $q = \exp(-\pi K'/K)$ does not exceed the square-root of $\frac{1}{2}$. This can readily be verified, either by differentiating the logarithmic derivative of Jacobi's

Proof. On (1), let $f(x)$ be a function of class C^∞ satisfying (2). Then $f(x)$ can be extended to an entire function $f(z)$ which, in addition, is of exponential type (S. Bernstein). Even if this additional information is not used, it follows that Taylor's expansion

$$f(x+iy) = \sum_{m=0}^{\infty} (iy)^m D^m f(x)/n! \tag{12}$$

is valid, at every x, for every y. Let x and y be real, and replace the interval (1) by the corresponding strip

$$-\tfrac{1}{2}\pi < \operatorname{Re} z < \tfrac{1}{2}\pi \qquad (z = x+iy). \tag{13}$$

Since the case $n=0$ of (2) implies that $f(z)$ is real for real z, it implies that $2\operatorname{Re} f(z)$ is the sum of $f(x+iy)$ and $f(x-iy)$. Hence, if y is replaced by $-y$ in (12) and if the result is added to (12) itself, it follows that

$$\operatorname{Re} f(z) = \sum_{m=0}^{\infty} y^{2m}(-D^2)^m f(x)/(2m)! \tag{14}$$

holds (by convergence) at every point of the z-plane and so, in particular, on the strip (13). But (14) and (2) imply that

$$\operatorname{Re} f(z) \geqq 0 \text{ on } (13). \tag{15}$$

Accordingly, if $f(x)$ satisfies (2), then it can be extended into an entire function $f(z)$ satisfying (15). Hence, more than what is needed for the completion of the proof is contained in the following criterion (*which does not assure that $f(z)$ is an entire function*):

Let $f(z)$ be a function which is regular on the strip (13) *and real-valued on the segment* (1), *where $z = x+iy$. Then $f(z)$ will satisfy* (15) *if and only if there exists on the closed t-interval $[-1,1]$ a monotone non-decreasing function $\phi(t)$ having the following property: If $p(w)$ is defined by the Stieltjes transform* (5) *on the simply-connected w-doman which results when both half-lines* (6) *are removed from the w-plane, then*

$$f(z) = p(\sin z)\cos z \text{ on } (13). \tag{16}$$

This criterion turns out to be a rephrasing of the *symmetric case* of (F. Riesz' version of) the Carathéodory-Toeplitz criterion for functions $F(w)$ which are regular on (4) and satisfy

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infinite product for $\operatorname{cn}(x,k)$ or by differentiating the pair of rational relations which connect the first derivatives of $\operatorname{cn}(x,k)$ and $\operatorname{sn}(x,k)$ with $\operatorname{sn}(x,k)$ and $\operatorname{cn}(x,k)$ themselves.

$$\text{Re}\, F(w) \geqq 0 \text{ on } (4) \tag{17}$$

(by the symmetry restriction is meant the condition

$$F(w^*) = F^*(w) \text{ on } (4), \tag{18}$$

where the asterisks denote complex conjugations).

Proof. The classical criterion, just mentioned, states that a function $F(w)$ which is regular on (4) will satisfy (17) if and only if there exists on the closed a-interval $[-\pi, \pi]$ a monotone non-decreasing function $\mu(a)$ in terms of which $F(w)$ is representable on (4) in the form

$$F(w) = \int_{-\pi}^{\pi} L(we^{ia})\, d\mu(a), \text{ where } L(s) = (1+s)/(1-s). \tag{19}$$

In order to characterize those particular distributions $d\mu(a) \geqq 0$ on $[-\pi, \pi]$ for which (19) will satisfy (18), expand (19) according to powers of w on (4) and insert the result in (18). Then comparison of like powers of w shows that (18) is equivalent to

$$2\int_{-\pi}^{\pi} e^{ina}\, d\mu(a) = 2\int_{-\pi}^{\pi} e^{-ina}\, d\mu(a), \text{ i.e., } \int_{-\pi}^{\pi} \sin na\, d\mu(a) = 0,$$

where $n = 1, 2, \cdots$, which in turn is equivalent to the restriction that $d\mu(a) = -d\mu(-a)$ if $0 < a < \pi$. If this is inserted in (19), a trivial reduction leads to

$$F(w) = \int_{0}^{\pi} P(w, \cos a)\, d\mu(a) \text{ on } (4), \tag{20}$$

where P *is Poisson's kernel*

$$P(w, \cos a) = (1-w^2)/(1-2w\cos a + w^2) \tag{21}$$

with complex w (instead of the standard $w = |w| = r$) and $\mu(a)$ is monotone non-decreasing on the closed a-interval $[0, \pi]$ but is otherwise arbitrary (the $\mu(a)$ of (20) is not the same as the $\mu(a)$ of (19) was). Thus (20), where $d\mu(a) \geqq 0$ on $[0, \pi]$, is necessary and sufficient in order that a function $F(w)$, regular on (4), be such as to satisfy both (17) and (18).

Let $f(z)$ be a function which is regular on (13), satisfies (15), and is real on (1), where $z = x + iy$. The latter restriction is equivalent to the symmetry assumption

$$f(z^*) = f^*(z) \text{ on } (13). \tag{22}$$

But it is a standard exercise that

$$w = \tan \tfrac{1}{2}z \tag{23}$$

is a Riemann mapping function belonging to the strip (14) and the circle (4). Since $(\tan \frac{1}{2}z)^* = \tan(\frac{1}{2}z^*)$, it follows that (17) and (18) are equivalent to (15) and (22) by virtue of $f(z) = F(w)$ and (23). Hence, by the end of the preceding paragraph, the class of functions $f(z)$ dealt with in the last italicized assertion is the class of those functions which are representable on (13) in the form

$$f(z) = \int_0^\pi P(\tan \tfrac{1}{2}z, \cos a)\, d\mu(a), \text{ where } d\mu(a) \geqq 0. \tag{24}$$

Finally, it is readily verified that, by virtue of (23), Poisson's complex kernel (21) becomes identical with the quotient of $\cos z$ and $1 - t \sin z$, where $t = \cos a$. Hence (24) can be written in the form

$$f(z) = \{ \int_0^\pi (1 - t \sin z)^{-1}\, d\mu(a) \} \cos z,$$

where $t = \cos a$ and $d\mu(a) \geqq 0$. Since this differs from (16) and (5) only in notation, the proof is complete.

Since the crucial formal point in the proof is the explicit simplification which results when (19) is reduced by the symmetry requirement (18), it is worth mentioning that, while the formulation of (19) in terms of the trigonometric moment problem supplied by spectral resolution of arbitrary unitary matrices in Hilbert space (cf., e. g., § 6, of E. Hopf's *Ergodentheorie*, Ergebnisse der Math., vol. 5 (1937), no. 2), the restriction (18) corresponds to the replacement of the unitary group by its orthogonal subgroup (cf. C. R. Putnam and A. Wintner, Amer. Jorn. of Math., vol. 74 (1952), pp. 52-78, and the Johns Hopkins thesis of M. H. Martin referred to there).

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VECTEURS ANALYTIQUES DANS LES REPRESENTATIONS DE GROUPES DE LIE.*

par P. CARTIER et J. DIXMIER.[1]

Introduction. Soit π une représentation continue[2] du groupe de Lie G dans l'espace de Banach $\mathfrak{H}$. Selon Harish-Chandra [1], on dit qu'un élément $\phi \in \mathfrak{H}$ est un *vecteur analytique* pour la représentation π si l'application $s \to \pi(s)\phi$ de G dans $\mathfrak{H}$ est analytique. Les vecteurs analytiques de $\mathfrak{H}$ forment un sous-espace vectoriel, non fermé en général, de $\mathfrak{H}$. Lorsque le groupe G est semi-simple et que la représentation π est "permise," Harish-Chandra a démontré dans [1] que les vecteurs analytiques sont partout denses dans $\mathfrak{H}$; nous nous proposons de généraliser ce résultat au cas d'un groupe de Lie quelconque et d'une représentation bornée sur un certain sous-groupe discrét. Le résultat précis est énoncé dans le théorème du § 6, qui est le résultat central de cet article.

Nous suivrons une méthode de démonstration analogue à celle de Harish-Chandra: alors que Harish-Chandra démontrait d'abord que les vecteurs analytiques sont partout denses dans le cas d'une représentation d'un groupe quasi-nilpotent, nous commencerons par une étude approfondie du cas des groupes résolubles; une fois obtenu le résultat pour les groupes résolubles, la démonstration suivra de très près celle de Harish-Chandra.

1. Notations. On désigne par $\boldsymbol{N}$ l'ensemble des entiers $\geqq 0$, par $\boldsymbol{Z}$ celui des entiers rationnels, par $\boldsymbol{R}$ le corps des nombres réels et par $\boldsymbol{C}$ le corps des nombres complexes.

L'élément neutre d'un groupe sera toujours noté e. Si f est une fonction sur un groupe G, la translatée à gauche ${}_sf\,(s \in G)$ est définie par ${}_sf(s') = f(ss')$. La translatée à droite f_s est définie par $f_s(s') = f(s's)$. Sur un groupe localement compact G, une mesure de Haar est une mesure $\geqq 0$ non nulle

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* Received November 4, 1957.

[1] Les numéros entre crochets renvoient à la Bibliographie située à la fin de l'article.

[2] La représentation π est dite continue si l'application $(x, \phi) \to \pi(x)\phi$ de $G \times \mathfrak{H}$ dans $\mathfrak{H}$ est continue, ou, ce qui revient au même, si pour tout $\phi \in \mathfrak{H}$, l'application $x \to \pi(x)\phi$ de G dans $\mathfrak{H}$ est continue.

invariante à gauche; l'élément d'intégrale correspondant sera noté ds si s désigne l'élément générique de G. Une *semi-norme* sur G est une fonction semi-continue inférieurement $\rho > 0$ vérifiant l'inégalité $\rho(ss') \leqq \rho(s)\rho(s')$ pour $s, s' \in G$; pour toute semi-norme ρ on notera $L^p_C(G, \rho)$ l'espace des fonctions numériques complexes sur G de puissance p^e intégrable ($p \geqq 1$) pour la mesure $\rho(s)ds$, muni de la norme définie par $\| f \|_{p,\rho} = \left(\int_G |f(s)|^p \rho(s)\, ds \right)^{1/p}$; la représentation régulière π de G dans $L^p_C(G, \rho)$ est définie par $\pi(s)f = {}_s f$ pour $f \in L^p_C(G, \rho)$.

Les espaces de Banach sont toujours supposés complexes (mais il n'y aurait pratiquement rien de changé s'ils étaient supposés réels).

Si $\mathfrak{H}$ est un espace de Banach, on désigne par $\mathcal{L}(\mathfrak{H})$ l'espace de Banach des endomorphismes continus de $\mathfrak{H}$. Si ρ est une semi-norme sur le groupe localement compact G et π une représentation continue de G dans l'espace de Banach $\mathfrak{H}$ telle que $\| \pi(s) \| \leqq \rho(s)$ pour tout $s \in G$, on pose

$$\pi(f) = \int_G f(s)\pi(s)\, ds \quad \in \mathcal{L}(\mathfrak{H})$$

pour $f \in L^1_C(G, \rho)$.

Tous les groupes de Lie envisagés sont supposés réels, sauf mention du contraire.

Soit G un groupe de Lie, d'algèbre de Lie $\mathfrak{g}$; on note exp l'application exponentielle de $\mathfrak{g}$ dans G, et pour toute représentation continue π de G dans un espace vectoriel de dimension finie, on note $d\pi$ la représentation correspondante de $\mathfrak{g}$. On a alors, pour tout $S \in \mathfrak{g}$

$$\pi(\exp S) = \exp d\pi(S) \tag{1}$$

où le second membre est défini comme d'habitude par $\exp T = \sum_{n \geqq 0} T^n/n!$. Soient G_1 et G_2 des sous-groupes de G, on écrira $G = G_1 \cdot G_2$ si et seulement si les conditions suivantes sont remplies:

1) G_1 et G_2 sont des sous-groupes analytiques de G;

2) l'application $(s_1, s_2) \to s_1 s_2$ de $G_1 \times G_2$ dans G est un isomorphisme de la première variété analytique sur la seconde.

2. Rappel de certains lemmes.

LEMME 1. *Soient E un espace localement compact, μ une mesure $\geqq 0$ sur E, $\mathfrak{H}$ l'espace de Banach des fonctions numériques complexes μ-intégrables. V une variété analytique complexe et f une fonction numérique complexe continue sur $V \times E$ satisfaisant aux conditions suivantes:*

(i) *il existe une fonction μ-intégrable $g \geqq 0$ sur E telle que l'on ait $|f(z,x)| \leqq g(x)$ pour tout $z \in V$ et tout $x \in E$;*

(ii) *pour tout $x \in E$, la fonction $z \to f(z,x)$ est holomorphe sur V.*

Si l'on note alors $\psi(z) \in \mathfrak{H}$ la fonction $x \to f(z,x)$, l'application $z \to \psi(z)$ de V dans $\mathfrak{H}$ est holomorphe.

C'est le lemme 23 de [1].

LEMME 2. *Soient G un groupe de Lie réel, $\mathfrak{H}$ un espace de Banach, π une représentation continue de G dans $\mathfrak{H}$ et ψ un élément de $\mathfrak{H}$. On suppose que l'application $s \to \pi(s)\psi$ est analytique en e; alors le vecteur ψ est analytique pour la représentation π.*

Cf. [1], p. 209, lignes 7-11.

LEMME 3. *Soient G un groupe de Lie réel, $\mathfrak{H}$ un espace de Banach, π une représentation continue de G dans $\mathfrak{H}$. Soit ρ une semi-norme sur G telle que $\|\pi(s)\| \leqq \rho(s)$ pour tout $s \in G$. Alors, si les vecteurs de $L^1_C(G,\rho)$ analytiques pour la représentation régulière sont partout denses dans $L^1_C(G,\rho)$, les vecteurs de $\mathfrak{H}$ analytiques pour π sont partout denses dans $\mathfrak{H}$.*

Soit A l'ensemble des éléments de $L^1_C(G,\rho)$ analytiques pour la représentation régulière. Soit $\mathfrak{K}$ l'ensemble des combinaisons linéaires de vecteurs $\pi(f)x$, où $x \in \mathfrak{H}$ et $f \in A$. Les éléments de $\mathfrak{K}$ sont analytiques pour π ([1], lemme 24), et il suffit de prouver que $\mathfrak{K}$ est partout dense dans $\mathfrak{H}$, autrement dit, que si ϕ est une forme linéaire continue sur $\mathfrak{H}$ nulle sur $\mathfrak{K}$, on a $\phi = 0$. Or, pour toute $f \in A$ et tout $x \in \mathfrak{H}$, on a

$$0 = \phi(\pi(f)x) = \int_G \phi(\pi(s)x)f(s)\,ds = \int_G \phi(\rho(s)^{-1}\pi(s)x)f(s)\rho(s)\,ds.$$

Puisque A est partout dense dans $L^1_C(G,\rho)$, on en déduit que $\phi(\rho(s)^{-1}\pi(s)x) = 0$ localement presque partout sur G, donc que $0 = \phi(\pi(e)x) = \phi(x)$ puisque la fonction $s \to \phi(\pi(s)x)$ est continue. D'où notre assertion (cf. aussi [1], lemme 25).

LEMME 4. *Soient X et Y deux variétés analytiques réelles, Y étant compacte, et f une application analytique de $X \times Y$ dans un espace de Banach $\mathfrak{H}$. Tout point de X possède alors un voisinage ouvert V ayant les propriétés suivantes:*

(i) *il existe un système de coordonnées analytiques $(t_1, \cdots, t_n)$ dans V;*

(ii) *il existe des applications analytiques f_e de Y dans $\mathfrak{H}$*

$$(e = (e_1, \cdots, e_n) \in \mathbf{N}^n)$$

telles que pour $x \in V$ et $y \in Y$, on ait:

$$f(x,y) = \sum_{e \in \mathbf{N}^n} f_e(y)\, t_1^{e_1}(x) \cdots t_n^{e_n}(x) \tag{2}$$

cette somme étant absolument et uniformément convergente dans $V \times Y$.

Il suffit de reprendre pratiquement sans changement, la démonstration de [1], de la ligne 20 le la p. 221 à la ligne 19 de la p. 222, en remplaçant partout G par X et K par Y.

3. Structure des groupes résolubles. Soient $\mathfrak{h}$ une algèbre de Lie résoluble réelle et $\mathfrak{n}$ un idéal nilpotent de $\mathfrak{h}$ contenant $[\mathfrak{h}, \mathfrak{h}]$; on note k la dimension de $\mathfrak{n}$ et n celle de $\mathfrak{h}$. Comme toute algèbre de Lie résoluble contient un idéal de codemnsion 1 et que tout sous-espace de $\mathfrak{h}$ contenant $\mathfrak{n}$ est un idéal de $\mathfrak{h}$, il existe une suite de sous-espaces $\mathfrak{h}_i$ de $\mathfrak{h}$ ayant les propriétés suivantes:

a) $\mathfrak{h}_i$ est de dimension i pour $0 \leqq i \leqq n$;

b) $\mathfrak{h}_i$ est un idéal de $\mathfrak{h}_{i+1}$ pour $0 \leqq i \leqq n-1$;

c) $\mathfrak{h}_k = \mathfrak{n}$.

La propriété a) nous permet de choisir une base $(X_1, \cdots, X_n)$ de $\mathfrak{h}$ telle que $(X_1, \cdots, X_i)$ soit une base de $\mathfrak{h}_i$. Ces données resteront fixées dans tout ce paragraphe.

LEMME 5. *Il existe un groupe de Lie réel H ayant $\mathfrak{h}$ pour algèbre de Lie et jouissant des propriétés suivantes:*

(i) *l'application $\phi: (t_1, \cdots, t_n) \to \prod_{i=1}^{n} \exp t_i X_i$ de $\mathbf{R}^n$ dans H est un isomorphisme de variété analytique;*

(ii) *soit $h \to (x_1(h), \cdots, x_n(h))$ l'application réciproque de ϕ et soit A l'algèbre complexe de fonctions sur H engendrée par 1, les x_i $(1 \leqq i \leqq n)$ et les $\exp \alpha x_j$ $(\alpha \in \mathbf{C}, k < j \leqq n)$; pour toute fonction $f \in A$, il existe des fonctions $f_p, f_p' \in A$ telles que l'on ait l'identité:*

$$f(hh') = \sum_{p=1}^{m} f_p(h) f_p'(h'). \tag{3}$$

Nous raisonnerons par récurrence sur n, le cas $n = 0$ étant trivial; nous supposerons donc $n > 0$ et nous poserons $\mathfrak{h}' = \mathfrak{h}_{n-1}$, $\mathfrak{n}' = \mathfrak{n} \cap \mathfrak{h}'$; la

dimension k' de $\mathfrak{n}'$ est égale à $n-1$ si $\mathfrak{n}=\mathfrak{h}$, et est égale à k si $\mathfrak{n}\subset\mathfrak{h}'$; on peut alors construire un groupe de Lie réel H', une application ϕ', une algèbre A' ayant les propriétés (i) et (ii) de l'énoncé, où l'on doit remplacer n par $n-1$ et k par k'.

A) Pour toute fonction $f\in A'$, le sous-espace vectoriel de A' engendré par les translatées à droite de f est de dimension finie sur $\boldsymbol{C}$ d'après la formule (3); par suite, A' est réunion de ses sous-espaces de dimension finie invariants par les translations à droite. Soient V' un tel sous-espace et π' la représentation de H' dans V' définie par $\pi'(h)\cdot f=f_h$.

Soit $(f_j)_{1\leqq j\leqq m}$ une base de V'; d'après la condition (ii) de l'énoncé il existe des fonctions $a_{lj}\in A'$ telles que:

$$\pi'(h)f_l=\sum_{j=1}^{m}a_{lj}(h)f_j \qquad (1\leqq l\leqq m); \tag{4}$$

mais, si $h=\exp tX_i$ avec $1\leqq i\leqq k'$, la définition des x_j' montre que $x_j'(h)=0$ pour $j>k'$ et $x_j'(h)=\delta_{ij}t$ (indice de Kronecker) pour $1\leqq j\leqq k'$. D'après la définition de A', on voit que pour tout $f\in A'$, $f(\exp tX_i)$ est un polynome en t pour $1\leqq i\leqq k'$; d'après la formule (4), la matrice de $\pi'(\exp tX_i)$ par rapport à la base (f_j) est donc un polynome en t puisque $a_{ij}\in A'$; mais d'après la formule (1), on a

$$\pi'(\exp tX_i)=\exp d\pi'(tX_i)=\sum_{r\geqq 0}(d\pi'(X_i))^r t^r/r!$$

ce qui impose que $d\pi'(X_i)$ soit nilpotent pour $1\leqq i\leqq k'$.

Soit $(V'_r)_{0\leqq p\leqq m}$ une suite de Jordan-Hölder de V' pour la représentation π' de H'; d'après le théorème de Lie sur les groupes résolubles, les espaces V'_p/V'_{p-1} sont de dimension 1. Pour $1\leqq i\leqq k'$, l'opérateur nilpotent $d\pi'(X_i)$ induit un opérateur nul dans V'_p/V'_{p-1}, autrement dit applique V'_p dans V'_{p-1}. Supposons la base (f_j) adaptée à la suite (V'_p), c'est-à-dire que V'_p admet $(f_1,f_2,\cdots,f_p)$ pour base; alors la matrice représentant $d\pi'(X_i)$ est de diagonale nulle pour $1\leqq i\leqq k'$. Il en résulte que si $Y(u)=X_i+\sum_j u_jX_j$ $(1\leqq j\leqq k';\ 1\leqq i\leqq n-1)$, la matrice qui représente $d\pi'(Y(u))$ est triangulaire et que ses termes non diagonaux sont linéaires en $u_1,\cdots,u_{k'}$; si $1\leqq i\leqq k'$, ses termes diagonaux sont nuls.[2a] Du lemme A de l'Appendice résulte que les coefficients de la matrice associée à l'opérateur $\pi'(\exp tY(u))=\exp(td\pi'(Y(u)))$ sont de la forme

$$\sum_{\alpha}E_\alpha(t)P_\alpha(u_1,\cdots,u_{k'}) \tag{5}$$

[2a] On notera que dans vous les cas les termes diagonaux de $d\pi'(Y(u))$ sont indépendants $u_1,\cdots,u_{k'}$.

où les P_α sont des polynomes et où les E_α sont des exponentielles-polynomes[3] *et même des polynomes si $1 \leqq i \leqq k'$.*

Or, de la formule (4) résulte $f_i(h) = \sum_{j=1}^{m} a_{ij}(h) f_j(e)$ et par suite, f_i est de la forme (5) ; comme A' est réunion de sous-espaces analogues à V', on voit que, pour tout $f \in A'$, la fonction

$$(t, u) \to f(\exp t(X_i + \sum_{j=1}^{k'} u_j X_j))$$

est du type (5).

B) Soit D la dérivation $Y \to [X_n, Y]$ de $\mathfrak{h}'$ et soit $\sigma(v)$ l'automorphisme $\exp vD$ de $\mathfrak{h}'$; comme D applique $\mathfrak{h}'$ dans $[\mathfrak{h}, \mathfrak{h}] \cap \mathfrak{h}' \subset \mathfrak{n} \cap \mathfrak{h}' = \mathfrak{n}'$, on a $\sigma(v) \cdot Y - Y \in \mathfrak{n}'$ pour tout $Y \in \mathfrak{h}'$, et les coefficients de $\sigma(v) \cdot Y - Y$ par rapport à la base $(X_1, \cdots, X_{k'})$ de $\mathfrak{n}'$ sont des exponentielles-polynomes en v et même des polynomes si $\mathfrak{n} = \mathfrak{h}$.[3a] De ce qu'on a démontré en A), il résulte alors que pour toute fonction $f \in A'$, la fonction $(t_i, v) \to f(\exp t_i \sigma(v) \cdot X_i)$ est de la forme :

$$\sum_{\alpha} E_\alpha(t_i) E_\alpha'(v) \tag{6}$$

où les E_α sont des exponentielles-polynomes et même des polynomes si $1 \leqq i \leqq k'$, tandis que les E_α' sont des exponentielles-polynomes et même des polynomes si $\mathfrak{n} = \mathfrak{h}$.

Soit $\tau(v)$ l'automorphisme du groupe simplement connexe H' associé à l'automorphisme $\sigma(v)$ de $\mathfrak{h}'$; pour $h = \prod_{i=1}^{n-1} \exp t_i X_i$, on a :

$$\tau(v) \cdot h = \prod_{i=1}^{n-1} \exp t_i \sigma(v) \cdot X_i. \tag{7}$$

De la définition de A', de la formule (6) et de la formule analogue à la formule (3) pour $n-1$ facteurs (que se déduit de (3) par récurrence immédiate), on déduit finalement que, *pour toute fonction $f \in A'$, la fonction $(h, v) \to f(\tau(v) \cdot h)$ est de la forme :*

$$\sum_{\alpha} E_\alpha(v) f_\alpha(h) \tag{8}$$

où $f_\alpha \in A'$ et les E_α sont des exponentielles-polynomes. Si $k = n$ (c'est-à-dire $\mathfrak{n} = \mathfrak{h}$), *les E_α sont même des polynomes.*

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[3] On appelle exponentielle-polynome à p variables $u_1, \cdots, u_p$ réelles ou complexes, une fonction de la forme : $\sum_i P_i(u_1, \cdots, u_p) \exp \sum_j a_{i,j} u_j$ où les P_i sont des polynomes à coefficients complexes et les $a_{i,j}$ des constantes complexes.

[3a] On notera que dans ce dernier cas, la dérivation D est nilpotente.

C) On peut maintenant former le produit semi-direct H de H' et $\boldsymbol{R}$ au moyen du groupe à un paramètre $\tau(v)$ d'automorphismes de H'. On rappelle que ce produit semi-direct est, en tant que variété, identique à $H' \times \boldsymbol{R}$ et que la loi de groupe est donnée par:

$$(h, v)(h', v') = (h \cdot \tau(v)(h'), v + v'). \tag{9}$$

Si l'on identifie X_n au générateur infinitésimal du groupe à un paramètre $v \to (e, v)$ de H, on voit immédiatement que $\mathfrak{h}$ s'identifie à l'algèbre de Lie de H, H' étant identifié au groupe des (h, e).

On a alors:

$$\phi(t_1, \cdots, t_n) = (\phi'(t_1, \cdots, t_{n-1}), e)(e, t_n) = (\phi'(t_1, \cdots, t_{n-1}), t_n)$$

ce qui prouve que ϕ est un isomorphisme de variété analytique. De plus, d'après la définition des x_i, on a:

$$x_i(h, v) = x_i'(h), \quad 1 \leqq i < n; \qquad x_n(h, v) = v$$

et par suite A est l'ensemble des fonctions de la forme (8) (où les E_α sont des exponentielles-polynomes en général, des polynomes si $\mathfrak{h} = \mathfrak{n}$). La formule (3) pour $f \in A$ résulte alors immédiatement de la définition de la loi de composition dans H (cf. (9)) et de la conclusion de B).—c. q. f. d.

Nous allons, en vue de besoins ultérieurs, déterminer la mesure de Haar sur le groupe H. L'automorphisme $\tau(v)$ de H' transforme une mesure de Haar μ sur H' en $\Delta(v)\mu$, où $\Delta(v)$ est le déterminant de l'automorphisme contragrédient de $\sigma(v)$; on a donc

$$\Delta(v) = (\det \sigma(v))^{-1} = \exp - v\mathrm{Tr}(D).$$

Si ν est la mesure de Haar usuelle sur $\boldsymbol{R}$, la formule (9) montre que la translation à gauche par (h, v) transforme la mesure μ en la mesure $\Delta(v)\mu \otimes \nu$; la mesure $\Delta(v)\, d\mu(h) d\nu(v)$ est donc une mesure de Haar sur H. Raisonnant par récurrence sur n, on voit que la mesure de Haar sur H est $\exp(\alpha_{k+1}x_{k+1} + \cdots + \alpha_n x_n)\, dx_1 \cdots dx_n$ en posant $\alpha_i = -\mathrm{Tr}(\mathrm{ad}\, X_i)$.

4. Vecteurs analytiques. Cas des groupes résolubles.

LEMME 6. *Soit N un groupe de Lie réel résoluble, connexe et simplement connexe. Soit ρ une semi-norme sur N. Les vecteurs de $L^1_C(N, \rho)$ analytiques pour la représentation régulière sont partout denses dans $L^1_C(N, \rho)$.*

A) Comme deux groupes de Lie connexes et simplement connexes ayant des algèbres de Lie isomorphes sont isomorphes, on peut d'après le lemme 5 trouver des fonctions $x_1, \cdots, x_n$ sur N ayant les propriétés suivantes:

a) l'application $h \to (x_1(h), \cdots, x_n(h))$ de N dans $\boldsymbol{R}^n$ est un isomorphisme de variété analytique;

b) il existe des exponentielles-polynomes P_i telles que:

$$(10) \quad x_i(hh') = P_i(x_1(h), \cdots, x_n(h);\ x_1(h'), \cdots, x_n(h')) \quad (1 \leqq i \leqq n);$$

c) la mesure de Haar sur N est de la forme $(\exp \sum \alpha_i x_i)\, dx_1 \cdots dx_n$. De plus, comme d'après (3) l'algèbre A du lemme 5 est réunion de ses sous-espaces de dimension finie invariants par les translations à gauche, on peut trouver des fonctions *réelles* y_j et a_{jk} $(1 \leqq j, k \leqq m)$ sur N, s'exprimant comme exponentielles-polynomes en les x_i et telles que:

d) $m \geqq n$ et $x_j = y_j$ pour $1 \leqq j \leqq n$;

e) on ait les identités:

$$(11) \quad y_j(hh') = \sum_{k=1}^{m} a_{jk}(h)\, y_k(h').$$

Comme les exponentielles-polynomes se prolongent en des fonctions entières de variables complexes, on peut d'après le principe du prolongement des identités, plonger N dans un groupe de Lie complexe N^c, les fonctions x_i, y_j et a_{jk} se prolongeant en des fonctions holomorphes sur N^c de sorte que les x_i forment un système de coordonnées complexes sur N^c et que les identités (10) et (11) soient valables sur N^c.

Enfin, pour achever ces préliminaires, notons que sur $\boldsymbol{R}$, toute seminorme est majorée par une exponentielle: $\rho(x) \leqq M \exp A \mid x \mid$ $(M, A \in \boldsymbol{R})$ (cf. par exemple [2], chap. VI). Revenant à la définition des x_i (lemme 5) et utilisant l'inégalité $\rho(\prod_{i=1}^{n} \exp t_i X_i) \leqq \prod_{i=1}^{n} \rho(\exp t_i X_i)$ on déduit la majoration:

$$(12) \quad \rho(h) \leqq M \exp A \sum_{i=1}^{n} |\, x_i(h) \,|, \qquad (h \in N)$$

pour une constante réelle A convenable et $M > 0$.

B) Soient Ω^c l'espace vectoriel complexe des formes quadratiques en des variables $Y_1, \cdots, Y_m$ à coefficients complexes, Ω le sous-espace réel formé des formes quadratiques à coefficients réels et Ω^+ le cône ouvert de Ω formé des formes quadratiques positives et non dégénérées. Nous définirons une représentation holomorphe du groupe opposé au groupe complexe N^c dans Ω^c en posant:

$$(13) \quad (\pi(h)F)(Y_1, \cdots, Y_m) = F(\sum_k a_{1k}(h) Y_k, \cdots, \sum_k a_{mk}(h) Y_k)$$

pour $F \in \Omega^c$ et $h \in N^c$.

Comme Ω^+ est ouvert dans Ω, pour h appartenant à un certain voisinage ouvert V de e dans N^c et $F \in \Omega^+$, la forme $\Re(\pi(h)F) - \frac{1}{2}F$ appartient à Ω^+ ($\Re$ désignant la partie réelle). On en déduit, pour $h \in V$, l'inégalité:

$$\Re F({}_h(y_1), \cdots, {}_h(y_m)) \geqq \tfrac{1}{2}F(y_1, \cdots, y_m)$$

d'où, si $f = \exp - F(y_1, \cdots, y_m)$:

$$|{}_h f| = \exp - \Re F({}_h(y_1), \cdots, {}_h(y_m)) \leqq \exp - \tfrac{1}{2}F(y_1, \cdots, y_m).$$

Or, puisque $F \in \Omega^+$, il existe une constante réelle $\alpha > 0$ telle que la forme $F(Y_1, \cdots, Y_m) - \alpha(Y_1^2 + \cdots + Y_n^2)$ soit positive; pour $h \in V$, on aura alors:

$$|{}_h f| \leqq \exp - \tfrac{1}{2}\alpha(x_1^2 + \cdots + x_n^2). \tag{14}$$

D'après la formule (12), la formule (14) et la forme de la mesure de Haar sur N, on a donc ${}_h f \in L^1_C(N, \rho)$ pour $h \in V$; de plus, pour h' fixé dans N, ${}_h f(h') = f(hh')$ dépend holomorphiquement de $h \in V$. L'inégalité (14) et le lemme 1 montrent que l'application $h \to {}_h f$ de V dans $L^1_C(N, \rho)$ est holomorphe, de sorte que f est un vecteur analytique de $L^1_C(N, \rho)$ pour la représentation régulière (lemme 2).

C) Soit $\mathfrak{F}$ l'ensemble des fonctions sur N de la forme $\exp - F(y_1, \cdots, y_m)$ avec $F \in \Omega^+$ et soit $\mathfrak{G}$ l'ensemble des combinaisons linéaires complexes d'éléments de $\mathfrak{F}$. Pour démontrer le lemme, il reste à faire voir que $\mathfrak{G}$ est partout dense dans $L^1_C(N, \rho)$, ce que nous ferons par une méthode un peu différente de celle de [1], pp. 218-220. Observons pour cela que la famille $\mathfrak{F}$ a les propriétés suivantes:

1) Les fonctions de $\mathfrak{F}$ sont continues, partout > 0 sur N, et nulles à l'infini d'après (14).

2) Le produit de deux fonctions de $\mathfrak{F}$ est dans $\mathfrak{F}$.

3) La famille $\mathfrak{F}$ est invariante par les translations à gauche, car si $f = \exp - F(y_1, \cdots, y_m)$, on a ${}_h f = \exp - (\pi(h)F)(y_1, \cdots, y_m)$; mais pour $h \in N$, la matrice $\| a_{jk}(h) \|$ est réelle et par suite $\pi(h)F \in \Omega^+$ si $F \in \Omega^+$.

4) $\mathfrak{F}$ sépare les points de N: raisonnant par l'absurde, supposons qu'il existe $h_0, h_1 \in N$ avec $h_1 \neq e$ et $f(h_0 h_1) = f(h_0)$ pour toute $f \in \mathfrak{F}$; d'après la propriété précédente, on aurait pour $f \in \mathfrak{F}$ et $h \in N$:

$$f(hh_1) = {}_{hh_0^{-1}} f(h_0 h_1) = {}_{hh_0^{-1}} f(h_0) = f(h)$$

d'où $f(h) = f(hh_1^m)$ pour toute $f \in \mathfrak{F}$ et tout entier m. Mais lorsque m tend

vers l'infini, il en est de même de $h_1{}^m$ [4]; comme f est nulle à l'infini sur N, on en déduit $f = 0$, ce qui est absurde.

Alors, $\mathfrak{G}$ est une algèbre de fonctions continues complexes sur N nulles à l'infini, séparant les points et stable par conjugaison. D'après le théorème de Stone-Weierstrass, toute fonction complexe continue nulle à l'infini sur N est limite uniforme de fonctions de $\mathfrak{G}$; soit alors f une fonction continue à support compact sur N; pour toute $f_0 \in \mathfrak{F}$ et tout $\epsilon > 0$, il existe $f_1 \in \mathfrak{G}$ telle que $|ff_0^{-1} - f_1| \leqq \epsilon$, d'où $\int_N |f - f_0 f_1| \rho\, dh \leqq \epsilon \int_N \rho f_0\, dh$. Comme $f_0 f_1$ appartient à $\mathfrak{G}$, on a montré que $\mathfrak{G}$ est partout dense dans $L^1_{\mathbf{C}}(N, \rho)$.

5. Un lemme de transition. Nous considérerons dans ce paragraphe la situation suivante:

G est un groupe de Lie connexe, K et N deux sous-groupes analytiques de G tels que $G = K \cdot N$; enfin Z est un sous-groupe central de G contenu dans K tel que $K^* = K/Z$ soit compact.

π est une représentation continue de G dans un espace de Banach $\mathfrak{H}$

μ est une représentation continue de K dans $\mathfrak{H}$ telle que:

a) quels que soient $s \in K$ et $s' \in G$, on a $\mu(s)\pi(s') = \pi(s')\mu(s)$;

b) l'application μ de K dans $\mathfrak{L}(\mathfrak{H})$ est analytique (au sens de la norme de $\mathfrak{L}(\mathfrak{H})$);

c) les restrictions de π et μ à Z sont identiques.

Soit $u \to u^*$ l'application canonique de K sur K^*; d'après c), l'opérateur $\mu(s)^{-1}\pi(s)$ est égal à 1 pour $s \in Z$, donc il existe une représentation continue π^* de K^* dans $\mathfrak{H}$ telle que $\mu(s)^{-1}\pi(s) = \pi^*(s^*)$ pour $s \in K$.

Ceci étant posé, on a le lemme suivant:

LEMME 7. *Soit f une fonction analytique sur K^* et soit ψ un élément de $\mathfrak{H}$, vecteur analytique de $\mathfrak{H}$ pour la représentation de N induite par π. Le vecteur $\phi = \int_{K^*} f(u^*)\pi^*(u^*)\psi\, du^*$ est un vecteur analytique de $\mathfrak{H}$ pour la représentation π.*

Nous pouvons étendre sans difficulté les raisonnements de [1], pp. 221-

[4] Si $h \in N$ est différente de e, h^m tend vers l'infini pour $m \to \infty$: sinon, d'après [4], chap. VI, lemme 2, les h^m seraient contenus dans un sous-groupe compact de G non réduit à e; mais si $x_i(h) \neq 0$ et $x_{i+1}(h) = \cdots = x_n(h) = 0$, la construction récurrente de H montre que l'on a $x_i(h^m) = m x_i(h)$, ce qui ne reste pas borné pour $m \to \infty$.

224. Pour $x \in G$, on a :

$$\pi(x)\phi = \int_{K^*} f(u^*)\pi(x)\pi^*(u^*)\psi\, du^*.$$

Posons, pour $x \in G$ et $u \in K$, $xu = u_x s(x,u)$ où $u_x \in K$ et $s(x,u) \in N$. Si $v \in Z$, on a $x(uv) = u_x s(x,u)v = (u_x v)s(x,u)$ puisque v est dans le centre de G. Donc $(u_x)^*$, $s(x,u)$ et $u^{-1}u_x$ ne dépendent que de x et de u^*. Nous noterons ces éléments $u_x{}^*$, $s(x,u^*)$ et $\gamma(x,u^*)$. Les applications $(x,u^*) \to u_x{}^*$, $(x,u^*) \to s(x,u^*)$ et $(x,u^*) \to \gamma(x,u^*)$ sont analytiques et l'on a :

$$\pi(x)\pi^*(u^*) = \pi(x)\mu(u)^{-1}\pi(u) = \mu(u)^{-1}\pi(u_x)\pi(s(x,u))$$
$$= \mu(u)^{-1}\mu(u_x)\pi^*(u_x{}^*)\pi(s(x,u^*)) = \mu(\gamma(x,u^*))\pi^*(u_x{}^*)\pi(s(x,u^*)).$$

Donc : $\pi(x)\phi = \int_{K^*} f(u^*)\mu(\gamma(x,u^*))\pi^*(u_x{}^*)\pi(s(x,u^*))\psi\, du^*.$

Pour x fixé, les applications $u \to u_x$ et $u \to u_{x^{-1}}$ sont des applications analytiques réciproques l'une de l'autre de K dans K. Donc les applications $u^* \to u_x{}^*$ et $u^* \to u_{x^{-1}}{}^*$ sont des automorphismes réciproques de la variété analytique K^*. Posons $du_x{}^* = D(x,u^*)\,du^*$, où D est une fonction analytique sur $G \times K^*$. On a :

$$\pi(x)\phi = \int_{K^*} \pi^*(u^*)D(x^{-1},u^*)f(u_{x^{-1}}{}^*)\mu(\gamma(x,u_{x^{-1}}{}^*))\pi(s(x,u_{x^{-1}}{}^*))\psi\, du^*.$$

Posons : $\psi(x,u^*) = D(x^{-1},u^*)f(u_{x^{-1}}{}^*)\mu(\gamma(x,u_{x^{-1}}{}^*))\pi(s(x,u_{x^{-1}}))\psi$. D'après le choix de ψ, $\pi(s(x,u_{x^{-1}}{}^*))\psi$ dépend analytiquement de (x,u^*) ; l'application $(\alpha,\beta,T,\eta) \to \alpha\beta(T\cdot\eta)$ de $\boldsymbol{R} \times \boldsymbol{R} \times \mathcal{L}(\mathfrak{H}) \times \mathfrak{H}$ dans $\mathfrak{H}$ est multilinéaire et continue et $(x,u^*) \to \mu(\gamma(x,u_{x^{-1}}{}^*))$ est une application analytique de $G \times K^*$ dans $\mathcal{L}(\mathfrak{H})$. Il en résulte que $\psi(x,u^*)$ dépend analytiquement de $(x,u^*) \in G \times K^*$; comme K^* est compact, il existe d'après le lemme 4 un voisinage V de e dans G et un système de coordonnées analytiques $(t_1,\cdots,t_n)$ dans V tels que, pour $(x,u^*) \in V \times K^*$, on ait :

$$\psi(x,u^*) = \sum_{\boldsymbol{e}} \psi_{\boldsymbol{e}}(u^*)t_1{}^{e_1}(x)\cdots t_n{}^{e_n}(x), \qquad \boldsymbol{e} = (e_1,\cdots,e_n) \in \boldsymbol{N}^n$$

où les applications $\psi_{\boldsymbol{e}}$ de K^* dans $\mathfrak{H}$ sont continues ; la somme précédente converge uniformément et absolument dans $V \times K^*$. Alors :

$$\pi(x)\phi = \int_{K^*} \pi^*(u^*)\psi(x,u^*)\,du^*$$
$$= \sum_{\boldsymbol{e}} \{\int_{K^*} \pi^*(u^*)\psi_{\boldsymbol{e}}(u^*)\,du^*\}\, t_1{}^{e_1}(x)\cdots t_n{}^{e_n}(x)$$

cette dernière famille de vecteurs étant encore sommable. Donc $x \to \pi(x)\phi$ est analytique dans V, ce qui, avec le lemme 2, achève la démonstration.

6. Vecteurs analytiques. Cas général. Nous commencerons par démontrer un lemme concernant la structure des groupes de Lie réels généraux. Ce lemme est contenu dans les résultats d'Iwasawa [3].

LEMME 8. *Soit G un groupe de Lie réel connexe et simplement connexe. Il existe des sous-groupes fermés K et N de G, et un sous-groupe discret Z du centre de G, avec les propriétés suivantes:*

(i) $$G = K \cdot N;$$

(ii) *N est résoluble connexe et simplement connexe;*

(iii) *K est produit direct d'un groupe compact K' et d'un groupe abélien connexe et simplement connexe A;*

(iv) *$Z \subset A$ et A/Z est compact.*

Comme G est simplement connexe, il est produit semi-direct d'un groupe de Lie semi-simple connexe et simplement connexe S et d'un groupe de Lie résoluble connexe et simplement connexe R (utiliser une décomposition de Levi de l'algèbre de Lie de G et construire *a priori* un tel produit semi-direct, qui sera isomorphe à G, puisque deux groupes de Lie connexes et simplement connexes ayant des algèbres de Lie isomorphes sont isomorphes). Les groupes S et R sont des sous-groupes fermés de G, et R est le radical de G. L'application $(s, r) \to sr$ est un isomorphisme de la variété analytique $S \times R$ sur la variété analytique G.

Il existe d'autre part (cf. par exemple [1], p. 223) des sous-groupes fermés K et M de S et un sous-groupe discret Z^0 du centre de S, possédant les propriétés suivantes:

a) $$S = K \cdot M;$$

b) M est résoluble connexe et simplement connexe;

c) K est produit direct d'un groupe compact K' et d'un groupe abélien connexe et simplement connexe A contenant Z^0, avec A/Z^0 compact.

Soit Z' l'intersection des noyaux des représentations linéaires continues de dimension finie de S. On sait que Z' est contenu dans le centre de S, et que S/Z' a un centre fini, de sorte que Z' est d'indice fini dans le centre de S. Donc $Z = Z' \cap Z^0$ est d'indice fini dans Z^0 et par suite A/Z est compact. Soit ρ la représentation adjointe de G dans l'algèbre de Lie $\mathfrak{g}$ de G; sa restriction à S est une représentation linéaire continue de dimension finie de S; son noyau contient donc Z' et par suite, pour $s \in Z$, on a $\mathrm{Ad}_{\mathfrak{g}} s = 1$; donc Z est contenu dans le centre de G.

Soit $N = M \cdot R$; comme R est distingué dans G, N est un sous-groupe fermé de G. En outre N/R est isomorphe à M, donc résoluble, de sorte que N est résoluble; comme M et R sont simplement connexes et que $N = M \cdot R$, N est simplement connexe. Enfin, comme $G = S \cdot R$ et que $S = K \cdot M$, on a $G = K \cdot N$.—c. q. f. d.

Nous allons maintenant démontrer le résultat central de cet article.

THÉORÈME. *Soient G un groupe de Lie réel, $\mathfrak{H}$ un espace de Banach et π une représentation continue de G dans $\mathfrak{H}$. Les vecteurs de $\mathfrak{H}$ analytiques pour π sont partout denses dans chacun des trois cas suivants:*

1) *G est connexe et simplement connexe; si Z désigne le sous-groupe introduit dans le lemme 8, les opérateurs $\pi(z)$ pour $z \in Z$ sont scalaires.*

2) *G est connexe et simplement connexe, et l'ensemble des opérateurs $\pi(z)$ pour $z \in Z$, est borné dans $\mathfrak{L}(\mathfrak{H})$.*

3) *La représentation π est bornée, c'est-à-dire que l'ensemble des opérateurs $\pi(s)$ pour $s \in G$, est borné dans $\mathfrak{L}(\mathfrak{H})$.*

Lorsque G est *résoluble* connexe et simplement connexe, les vecteurs analytiques sont partout denses dans $\mathfrak{H}$, comme il résulte des lemmes 3 et 6.

Supposons G connexe et simplement connexe et introduisons les groupes K, A, N et Z du lemme 8; comme A est abélien connexe et simplement connexe, il est produit direct de deux sous-groupes A' et A'', Z étant contenu dans A'' et image de $\mathbf{Z}^p$ par un isomorphisme ϕ de $\mathbf{R}^p$ sur A'' ($p = \dim A''$).

Dans le cas 1), posons $\pi(z) = \chi(z) \cdot 1$ pour $z \in Z$; il est clair que l'homomorphisme χ de Z dans le groupe multiplicatif $\mathbf{C}^*$ des nombres complexes se prolonge en un homomorphisme continu χ' de A'' dans $\mathbf{C}^*$. Comme K est produit direct de K', A' et A'', on peut prolonger χ' en un homomorphisme χ'' de K dans $\mathbf{C}^*$ égal à 1 sur K' et A'; on posera $\mu(u) = \chi''(u) \cdot 1$ et l'homomorphisme μ de K dans le groupe des opérateurs continus inversibles de $\mathfrak{H}$, est une application analytique de K dans $\mathfrak{L}(\mathfrak{H})$.

Dans le cas 2), l'expression $\rho(s) = \sup \| \pi(sz) \|$, $z \in Z$, est finie pour tout $s \in G$ et représente une semi-norme sur G telle que $\rho(sz) = \rho(s)$ pour $z \in Z$ et $s \in G$; introduisons la représentation régulière π' de G dans $L^2_{\mathbf{C}}(G, \rho)$. Comme ρ est invariante par les translations par les éléments de Z, les opérateurs $\pi'(z)$ sont unitaires pour $z \in Z$; posons:

$$\pi'(\phi(e_1, \cdots, e_p)) = U_1^{e_1} \cdots U_p^{e_p} \qquad (e_1, \cdots, e_p \in \mathbf{Z})$$

où les U_i sont unitaires. On peut écrire $U_i = \exp H_i \sqrt{-1}$, où l'opérateur hermitien continu H_i commute à tout opérateur commutant à U_i; soit alors

μ la représentation de A'' dans $\mathfrak{H}$ definie par:

$$\mu(\phi(t_1, \cdots, t_p)) = \prod_{i=1}^{p} \exp \sqrt{-1}\, t_i H_i \qquad (t_1, \cdots, t_p \in \boldsymbol{R})$$

que nous prolongerons à K en posant $\mu(k'a'a'') = \mu(a'')$ pour $k' \in K'$, $a' \in A'$ et $a'' \in A''$. L'application μ de K dans $\mathfrak{L}(L^2_{\boldsymbol{C}}(G, \rho))$ est alors analytique et les opérateurs $\mu(u)$ commutent à tout opérateur commutant aux U_i, en particulier ils commutent aux $\pi'(s)$ $(s \in G)$. On a enfin $\mu(u) = \pi(u)$ pour $u \in Z$.

On notera alors, que pour tout groupe compact, les fonctions analytiques sont partout denses dans l'espace des fonctions intégrables pour la mesure de Haar, puisqu'il en est ainsi des coefficients des représentations continues de dimension finie. Le lemme 7 et ce qu'on a vu des groupes résolubles montrent alors que, dans le cas 1) les vecteurs analytiques sont partout denses dans $\mathfrak{H}$, et dans le cas 2) les vecteurs analytiques pour la représentation π' sont partout denses dans $L^2_{\boldsymbol{C}}(G, \rho)$. D'autre part, l'application $(f, g) \to fg$ est une application bilinéaire continue de $L^2_{\boldsymbol{C}}(G, \rho) \times L^2_{\boldsymbol{C}}(G, \rho)$ dans $L^1_{\boldsymbol{C}}(G, \rho)$; donc les vecteurs de $L^1_{\boldsymbol{C}}(G, \rho)$ analytiques pour la représentation régulière sont partout denses dans $L^1_{\boldsymbol{C}}(G, \rho)$. D'après le lemme 3, les vecteurs analytiques pour la représentation π sont donc partout denses dans $\mathfrak{H}$, dans le cas 2).

Dans le cas 3), soit G_0 la composante connexe de e dans G et soit G_0^* le revêtement universel de G_0; la représentation π de G définit de manière bien déterminée une représentation continue bornée π^* de G_0^*. D'après 2), les vecteurs de $\mathfrak{H}$ analytiques pour π^* sont partout denses dans $\mathfrak{H}$; mais comme G_0^* est localement isomorphe à G_0, un vecteur de $\mathfrak{H}$ analytique pour π^* est analytique pour la restriction de π à G_0 et donc pour π (lemme 2).—c.q.f.d.

Appendice.

LEMME A. *Soit $U = \| u_{ij} \|$ une matrice à coefficients complexes, à n lignes et n colonnes; on suppose $u_{ij} = 0$ pour $i < j$; on considère les $u_{ii} = a_i$ comme des constantes et l'on note t une variable complexe. Dans ces conditions, les coefficients de la matrice $\exp tU$ sont de la forme*

$$\sum_{\alpha} E_\alpha(t) P_\alpha(u_{21}, u_{31}, \cdots, u_{n,n-1})$$

où les P_α sont des polynomes de degré $< n$ et les E_α sont tous de la forme

$$\sum_{i=1}^{n} e^{a_i t} Q_i(t), \tag{15}$$

Q_i étant un polynome de degré $< n$.

Soit $V(t) = \exp tU$. On sait que $dV(t)/dt = U \cdot V(t)$, d'où

$$\prod_{i=1}^{n} (d/dt - a_i) V(t) = \prod_{i=1}^{n} (U - a_i \cdot 1) \cdot V(t) = 0$$

puisque, d'après le théorème de Hamilton-Cayley, on a $\prod_{i=1}^{n} (U - a_i \cdot 1) = 0$. Les coefficients de $V(t)$ sont donc, lorsque les u_{ij} $(i > j)$ sont fixés, solutions de l'équation differentielle $\prod_{i=1}^{n} (d/dt - a_i) f(t) = 0$, donc de la forme (15).

Considérons maintenant t comme fixé. On peut décomposer de manière unique U en somme d'une matrice diagonale D et d'une matrice N avec $n_{ij} = 0$ pour $i \leqq j$ (et donc $n_{ij} = u_{ij}$ pour $i > j$). Soit (e_i) la base canonique de $\boldsymbol{C}^n$, et considérons D et N comme des endomorphismes de $\boldsymbol{C}^n$; si V_i est le sous-espace de $\boldsymbol{C}^n$ de base $(e_i, e_{i+1}, \cdots, e_n)$, on a $D(V_i) \subset V_i$ et $N(V_i) \subset V_{i+1}$, il en résulte que tout monôme (non commutatif) en D et N dont le degré en N est $\geqq n$ est nul; par suite, pour tout entier m, les coefficients de $(D+N)^m$ sont, pour D fixé, des polynomes de degré $< n$ en les u_{ij} $(i > j)$; il en résulte que, pour t et D fixés, les coefficients de $\exp t(D+N) = \sum_{m \geqq 0} t^m (D+N)^m / m!$ sont des polynomes de degré $< n$ en les u_{ij} $(i > j)$:

Ceci posé, le lemme A est une conséquence du lemme facile suivant, dont on laisse la démonstration au lecteur:

LEMME B. *Soient X et Y deux ensembles, K un corps, V (resp. W) un espace vectoriel de dimension finie de fonctions sur X (resp. Y). Soit f une application de $X \times Y$ dans K telle que:*

a) *pour tout $x \in X$, l'application $y \to f(x,y)$ est dans W;*

b) *pour tout $y \in Y$, l'application $x \to f(x,y)$ est dans V.*

Alors f est de la forme $f(x,y) = \sum_{\alpha} g_\alpha(x) h_\alpha(y)$ avec $g_\alpha \in V$ et $h_\alpha \in W$.

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BIBLIOGRAPHIE.

[1] Harish-Chandra, "Representations of a semi-simple Lie group on a Banach space I," *Transactions of the American Mathematical Society*, vol. 75 (1953), pp. 185-243.

[2] E. Hille, *Functional analysis and semi-groups*, American Mathematical Society Colloquium Publications, vol. 31, New York, 1948.

[3] Iwasawa, "On some types of topological groups," *Annals of Mathematics*, vol. 50 (1949), pp. 507-558.

[4] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, 2ème édition, *Actualités Scientifiques et Industrielles*, no. 1154, Paris, Hermann.

THE PROBLEM OF MINIMAL MODELS IN THE THEORY OF ALGEBRAIC SURFACES.*

By Oscar Zariski.†

Introduction. The non-singular varieties, in a given birational class of varieties (over an algebraically closed field k), form—in a well-known fashion—a partially ordered set $\mathfrak{B}$. Any minimal element of $\mathfrak{B}$ is called a *relatively minimal model*, and the lower bound of $\mathfrak{B}$ (if it exists) is called a *minimal model* (we identify biregularly equivalent varieties). The object of the present paper is to prove, for ground fields of arbitrary characteristic, the following classical result (known to the Italian geometers in ths classical case):[1] *if a birational class $\mathfrak{B}$ of non-singular surfaces contains no minimal model then $\mathfrak{B}$ is the birational class of a ruled surface* (§ 3, Fundamental Theorem A). We prove actually a stronger result. It can be proved that if our birational class of non-singular surfaces contains no minimal model and if F is a relatively minimal model in the class (such models always exist even in the case of higher varieties, as long as our birational class contains non-singular models at all; see Corollary 1.8), then F carries an irreducible exceptional curve E of the second kind (Proposition 3.3, (a) and Proposition 2.6). It can also be shown that the self-intersection number (E^2) of an irreducible exceptional curve E of the second kind (on a non-singular surface F) is $\geqq 0$ (and that if E has singularities then $(E^2) > 0$); see Proposition 2.7. Then

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[1] See, for instance, F. Enriques [2], pp. 372-374. The proof in the classical case is due to Castelnuovo and Enriques, but that proof cannot be regarded as entirely rigorous. Furthermore, some steps of the proof, even those which could be accepted without objections in the classical case, would break down in the case of characteristic $p \neq 0$. Our proof differs from the classical proof also in some of its central ideas. Thus: (a) we do not have to presuppose that the Castelnuovo criterion of rationality $p_a = P_2 = 0$ has already been established (and in fact, the validity of that criterion does not result from our proof, except in some special cases; see, however, the concluding remarks of our introduction); (b) the crucial phases of the proof are based on specific properties of exceptional curves of the second kind.

what we actually prove is the following stronger result: *if a non-singular surface F carries an irreducible exceptional curve E of the second kind, then F is birationally equivalent to a ruled surface, and that if, furthermore, (E^2) is positive then F is a rational surface* (Fundamental Theorem B, § 3).

A preliminary study of the problem of minimal models is contained in our monograph "Introduction to the problem of minimal models in the theory of algebraic surfaces," now in course of publication in the "Publications of the Mathematical Society of Japan." In that monograph (based on a series of lectures given in Tokyo and Kyoto in the fall of 1956) we give an exposition of some background material concerning rational transformations and linear systems, then develop the theory of exceptional curves on a non-singular surface, and finally give an outline of the contents of the present paper. This monograph will be referred to as IP. Frequent and explicit references to various propositions proved in IP will be made in the present paper, especially in regard to the theory of exceptional curves.

In Section 1 we deal with varieties of any dimension and we prove that the class $\mathfrak{B}$ satisfies the descending chain condition (Proposition 1.5). Thus the existence of relatively minimal models as assured (as long as $\mathfrak{B}$ is non-empty; a question which is still open for varieties of dimension >2 in the case of characteristic $\neq 0$, and for varieties of dimension >3 in the case of characteristic zero). This result is a rather straightforward consequence of the theorem of Neron-Severi.[2] In Section 2 we review some of the results concerning exceptional curves which have been obtained in IP, and we derive a few other properties of exceptional curves. In Section 3 we give various criteria of existence of minimal models. In partilular, we show that if there does *not* exist a minimal model then we are dealing with surfaces all plurigenera of which are zero (Proposition 2.9). This result (above all, the vanishing of the geometric genus p_g and of the bigenus P_2) is repeatedly used in the rest of the paper. In Section 4 we prove the fundamental

[2] In the case of surfaces it is possible (as was pointed out by Serre) to prove the descending chain condition by using sheaf theory. Namely, one observes that if $F < F'$, where F and F' are non-singular surfaces, then $h^{1,1}(F) < h^{1,1}(F')$, where the notations are those of the theory of sheaves. This inequality is first proved directly in the case in which the birational transformation $F \to F'$ is locally quadratic, and then one uses our factorization theorem for antiregular birational transformations of non-singular surfaces (see IP, Theorem II. 1. 2; or also our paper [12], Lemma on p. 538).

A second proof (which we shall publish in the Mem. Col. Sci. of Kyoto University) is based on the consideration of the anticanonical system $|-K|$ of an algebraic surface. It can be shown that if we have an infinite strictly descending chain $F_1 > F_2 > \cdots$ of non-singular algebraic surfaces, then $\dim |-K(F_j)| \to +\infty$, and this can be shown to lead to a contradiction.

theorem B (stated in Section 3) under the assumption that $(E^2) = 0$ and also in the case in which E is non-singular. The conditions $p(E) = 0$ and $(E^2) \geqq 0$ characterize the irreducible *non-singular* exceptional curves of the second kind (Proposition 2.8), and what we are actually proving in Section 4 is that if a non-singular surface F carries an irreducible curve E such that $p(E) = 0$ and $(E^2) \geqq 0$, then F is birationally equivalent to a ruled surface.

The remainder of the paper deals with the proof of the fundamental theorem B in the case $(E^2) > 0$. In this case it is the rationality of F that has to be proved. A first step in that direction is the proof of the vanishing of the arithmetic genus p_a of F (Section 5). It will be noted that our present proof of the vanishing of p_a is much simpler than the proof which was outlined in IP (III, § 3). We now make no use of the concept of strongly minimal models (IP, Definition III. 3. 1), and—what is more important—we do not use a local result which was stated without proof in IP (Proposition III. 3. 4). The proof of this local result will be published elsewhere. What is common to both proofs is the recourse to Abelian varieties and the use of the equality $q = -p_a$, which holds on any surface of geometric genus zero.

The proof of the fundamental theorem in the case $(E^2) > 0$ (and under the permissible assumption that F is a relatively minimal model; see remark at the end of Section 5) has to be divided into various cases, according to the value of (K^2), where K is a canonical divisor on F. A very short proof, given in Section 6, settles the case $(K^2) \leqq 0$. In Section 7 we develop some general properties of surfaces which satisfy the Castelnuovo criterion $p_a = P_2 = 0$ *and* the inequality $(K^2) > 0$. In Section 8 we prove the Castelnuovo criterion of rationality in the case $(K^2) \geqq 3$. This leaves us only with two cases for completing the proof of the fundamental theorem: $(K_2) = 1$ and $(K^2) = 2$. The proof in the case $(K^2) = 1$ (Section 9) is very short and simple. In the last section (Section 10) we deal with the case $(K^2) = 2$, and here again, as in Section 8, we are proving not merely the fundamental theorem B but actually the Castelnuovo criterion of rationality $p_a = P_2 = 0$ in the case $(K^2) = 2$ (the latter implies the former once the vanishing of p_a has been established in Section 5). We first give a very simple proof, but unfortunately this proof does not work for characteristic 2. We then give a second (and much longer) proof which is valid for any characteristic. This second proof is constructive in the sense that we construct the entire family of algebraic surfaces with $p_a = P_2 = 0$ and $(K^2) = 2$. We show that the general member of this family is a rational surface, and we then prove the rationality of the given surface F by a specialization argument.

From the foregoing outline it is seen that the present investigation contains a proof of Castelnuovo's criterion of rationality of a surface ("a surface is rational if $p_a = P_2 = 0$") in the case $(K^2) \geqq 2$. A proof of Castelnuovo's criterion of rationality, in the classical case, was given by Kodaira in a lecture course at Princeton in 1956 (the existence of this proof, and the proof itself, because known to me in Paris through an exposé by Serre in a Bourbaki seminar in 1957). It appears clearly from that exposé that Kodaira's proof, *in the case* $(K^2) \leqq 0$, is algebraic in nature and is valid for any characteristic. [On the contrary, in the case $0 < (K^2) \leqq 9$, Kodaira's proof uses topological considerations; our proof in the case $(K^2) \geqq 3$ was presented in a Kyoto seminar in 1956 and is similar to, but somewhat simpler than, the proof of Kodaira in the case $(K^2) > 9$]. Thus the proof of Castelnuovo's criterion of rationality in the abstract case is complete, except for the case $(K^2) = 1$. We have a proof also in this case, but in order not to lengthen the present paper we do not include it here; it will be published elsewhere.

We point out that the proof of the fundamental theorem in the case $(K^2) \leqq 0$ which we gave in our Kyoto seminar was very long, while in the present paper the proof takes a few lines (Section 6). The reason for this is that in the Kyoto seminar we were essentially proving also the Castelnuovo criterion of rationality in the case $(K^2) \leqq 0$ (and not merely the fundamental theorem B). However, our Kyoto proof of Castelnuovo's criterion of rationality in the case $(K^2) \leqq 0$ was not as simple as Kodaira's proof and is therefore superseded by the latter.

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10. Proof of Castelnuovo's criterion of rationality in the case $(K^2) = 2$.

1. Minimal and relatively minimal models. Let k be an algebraically closed field and let Σ be a field of algebraic functions of r independent variables over k, i.e., a finitely generated extension of k, of transcendence degree r. By a *projective model* of Σ/k we mean a pair (V,P), where V is a (projective) variety which is defined and irreducible over k, P is a general point of V/k, and $\Sigma = k(P)$. In the sequel we shall often refer to V itself as a projective model of Σ/k and shall write V instead of (V,P) without fear of confusion.

We shall denote by $\mathfrak{B}^*$ the totality of all normal projective models of Σ/k, and by $\mathfrak{B}$ the totality of all non-singular projective models of Σ/k.

Given two projective models $V=(V,P)$ and $V'=(V',P')$ of Σ/k, there is a unique birational transformation T of V onto V' such that $T(P)=P'$. We shall say that V *dominates* V', or in symbols: $V \succ V'$, if T is regular on V. The relation $\succ$ defines a partial ordering of the birational class $\mathfrak{B}^*$. As a rule we shall identify biregularly equivalent models, i.e., models V and V' such that $V \succ V'$ *and* $V' \succ V$.

Let V, V' be members of $\mathfrak{B}^*$ such that $V \succ V'$ and let $T: V \to V'$ be the regular transformation of V onto V'. To any irreducible subvariety W/k of V/k there corresponds a unique irreducible subvariety W'/k of V'/k, and we have $\dim W' \leqq \dim W$. We say that W is an exceptional variety of T if $\dim W' < \dim W$. It is known (see IP, Corollary I. 2. 4) that there exists a subvariety E of V, called the *exceptional locus* of T, having the following properties: 1) E is defined over k; 2) every irreducible component of E/k is an exceptional variety of T; 3) every exceptional variety of T is contained in E. The exceptional locus E of T is identional with the set of all points of V at which T is not biregular.

Let $\mathfrak{G}$ denote the group of divisorial cycles on V which is generated by the prime divisorial cycles which are rational over k. Let $\mathfrak{G}'$ have the similar meaning for the variety V'. We shall now associate with the regular transformation $T: V \to V'$ a mapping, denoted by the same letter T, of $\mathfrak{G}$ into $\mathfrak{G}'$.

If Z is a prime divisorial cycle on V, rational over k, and if the variety Z is not exceptional for T, then the corresponding variety on V' is of the same dimension as Z and defines a prime divisorial cycle Z', rational over k. We then set $T(Z)=Z'$. If, however, Z is an exceptional variety we set $T(Z)=0$. We then extend the definition of the cycle transformation $Z \to T(Z)$ to the whole group $\mathfrak{G}$, by linearity. If we denote by $\mathfrak{G}(T)$ the subgroup of $\mathfrak{G}$ generated by prime divisorial cycles which are exceptional for T, then the cycle transformation T is a homomorphism of $\mathfrak{G}$ into $\mathfrak{G}'$, with kernel $\mathfrak{G}(T)$. It is clear that T is a mapping of $\mathfrak{G}$ onto $\mathfrak{G}'$, for every $(r-1)$-dimensional

irreducible subvariety W'/k of V'/k corresponds, under T, to at least one irreducible subvariety W/k of V/k.

Let $\mathfrak{G}_a$ (resp. $\mathfrak{G}'_a$) be the subgroup of $\mathfrak{G}$ (resp. of $\mathfrak{G}'$) which is generated by the cycles which are algebraically equivalent to zero.

PROPOSITION 1.1. *Denoting by* $\mathfrak{E}(T) + \mathfrak{G}_a$ *the subgroup of* $\mathfrak{G}$ *which is generated by the subgroups* $\mathfrak{E}(T)$ *and* $\mathfrak{G}_a$, *we have*

$$T(\mathfrak{E}(T) + \mathfrak{G}_a) = \mathfrak{G}'_a.$$

Proof. We shall use notations and results from Weil's paper [10]. Any cycle in $\mathfrak{G}_a$ is of the form $Z(N) - Z(M)$, where Z is a divisorial cycle on the product $V \times C$ of V by a non-singular irreducible curve C/k and where M and N are points of C which are rational over k (Weil [10], Lemma 10). Here the divisorial cycle Z may be assumed to be a "reduced" one, i.e., free from prime components of the form $V \times W'$ or $W \times C$ ($W' \in V'$, $W \subset V$). As to the symbol $Z(Q)$, where Q is a point of C, it is defined by the relation $Z(Q) \times Q = Z \cdot V \times Q$. Since Z is a reduced divisorial cycle and both V and C are normal, it is true for any point Q of C that all the prime components of the cycle $Z \cdot V \times Q$ are proper intersections of the variety of the cycle Z with the variety $V \times Q$, i.e., they are all of dimension $r-1$ and are simple for $V \times C$ (and also for $V \times Q$). They are therefore all of the form $\Gamma \times Q$, where Γ is a prime divisorial cycle on V. Thus $Z(Q)$ is a divisorial cycle on V.

The birational transformation $T^*: V \times C \to V' \times C$ defined $T^*(A \times B) = T(A) \times B$ ($A \in V, B \in C$) is regular. By the preceding definition, we have an induced transformation, denoted by the same letter T^*, of the group of divisorial cycles of $V \times C$ onto the group of divisorial cycles of $V' \times C$. We set $Z' = T^*(Z)$. We now observe that if Q is a rational point of C, then the restriction to $V \times Q$ of the birational transformation $T^*: V \times C \to V' \times C$ is a birational transformation $T_Q: V \times Q \to V' \times Q$, defined over k, and it is immediate that for any divisorial cycle Γ on V we have

$$T_Q(\Gamma \times Q) = T(\Gamma) \times Q. \tag{1}$$

We have the following relation:

$$T^*(Z) \cdot V' \times Q = T_Q(Z(Q) \times Q). \tag{2}$$

In fact, let Γ be any prime divisorial cycle on V, rational over k. If Γ is not an exceptional variety of T, then T is biregular at a general point of Γ/k, T^* is biregular at a general point of $\Gamma \times Q/k$. In this case, the coefficient of $\Gamma \times Q$ in $Z \cdot V \times Q$ is the same as the coefficient of $T(\Gamma) \times Q$ in

$T^*(Z)\cdot V'\times Q$. On the other hand, the coefficient of $\Gamma\times Q$ in $Z\cdot V\times Q$ is the same as the coefficient of $T(\Gamma)\times Q$ in $T_Q(Z\cdot V\times Q)$, by the definition of the cycle transformation T_Q. Hence in this case, the cycle $T(\Gamma)\times Q$ occurs with the same coefficient in both sides of (2). Now, assume that Γ is an execptional variety of T. In that case, $\Gamma\times Q$ is also an exceptional variety of T_Q [by (1)], and thus $T_Q(\Gamma\times Q)=0$. Since any prime divisorial cycle of $V'\times Q$ is the T_Q-transform of a cycle of the form $\Gamma\times Q$, where Γ is a non-exceptional prime divisor cycle of V, relation (2) is established.

We now go back to our cycle $Z(N)-Z(M)$ in $\mathfrak{G}_a$. Let $Z(M)=\sum a_i\Gamma_i$, $Z(N)=\sum b_j\Delta_j$, where Γ_i and Δ_j are prime divisorial cycles on V, rational over k. We have

$$Z\cdot V\times M=\sum a_i(\Gamma_i\times M),\qquad Z\cdot V\times N=\sum b_j(\Delta_j\times N).$$

Hence, by (2),

$$Z'\cdot V'\times M=\sum a_i(T(\Gamma_i)\times M),\qquad Z'\cdot V'\times N=\sum b_j(T(\Delta_j)\times N),$$

where $Z'=T^*(Z)$. Consequently

$$Z'(M)-Z'(N)=T(Z(M)-Z(N)). \tag{3}$$

This establishes the inclusion $T(\mathfrak{G}_a)\subset\mathfrak{G}'_a$ and therefore also the inclusion $T(\mathfrak{E}(T)+\mathfrak{G}_a)\subset\mathfrak{G}'_a$, since $\mathfrak{E}(T)$ is the kernel of T. The opposite inclusion $\mathfrak{G}'_a\subset T(\mathfrak{G}_a)$ follows from the fact that given any reduced divisorial cycle Z' on $V'\times C$ there exists a reduced divisorial cycle Z on $V\times C$ such that $Z'=T^*(Z)$. This completes the proof of the proposition.

COROLLARY 1.2. *The group of Neron-Severi of the variety V' (i.e., the group $\mathfrak{G}'/\mathfrak{G}'_a$) is isomorphic to $\mathfrak{G}/(\mathfrak{E}(T)+\mathfrak{G}_a)$ and is therefore a homomorphic image of the group of Neron-Severi of the variety V.*

COROLLARY 1.3. *If the exceptional locus E of T has dimensional $r-1$ then the number of Picard $\rho(V')$ of V' ($\rho(V')=$ rank of $\mathfrak{G}'/\mathfrak{G}'_a$) is less than the number of Picard $\rho(V)$ of V.*

For under our assumption, there exists a prime divisorial cycle Γ on V such that $T(\Gamma)=0$, and the corollary follows from the fact that no integral multiple $n\Gamma$ of Γ ($n\neq 0$) can be algebraically equivalent to zero.

COROLLARY 1.4. *If V is either a non-singular variety or is a normal surface, and if the regular transformation $T\colon V\to V'$ is not biregular, then $\rho(V)>\rho(V')$.*

For in either case the exceptional locus E of T has dimension $r-1$

(van der Waerden, *Algebraische Korrespondenzen und rationale Abbildungen*, Math. Ann. *110*, 1934, p. 154; Zariski, *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc., *53*, 1943, p. 532, Corollary to Theorem 17).

As a consequence of Corollary 1.4 and the finiteness of the number ρ of Picard we have

PROPOSITION 1.5. *The birational class $\mathfrak{B}$ of non-singular projective models of Σ/k satisfies the descending chain condition. If $r=2$ then also the birational class $\mathfrak{B}^*$ (of normal projective models of Σ/k) satisfies the descending chain condition.*

Note that in this proposition it is tacitly assumed that biregularly equivalent varieties have been identified.

DEFINITION 1.6. *By a relatively minimal model of Σ/k we mean a minimal element o fthe ordered set $\mathfrak{B}$, i.e., a non-singular projective model V of Σ/k which does not dominate (in the strict sense) any other non-singular projective model of Σ/k.*

DEFINITION 1.7. *By a minimal model of Σ/k we mean a lower bound of the ordered set $\mathfrak{B}$, i.e., a non-singular projective model V of Σ/k such that $V \prec V'$ for all non-singular projective models V' of Σ/k.*

COROLLARY 1.8. *If a field Σ/k of algebraic functions of r independent variables possesses at all non-singular projective models (in particular, if $r=2$, or if $r=3$ and the characteristic of k is 0) then Σ/k also possesses a relatively minimal model, and, in fact, given any non-singular projective model V of Σ/k there exists a relatively minimal model V' of Σ/k such that $V' \prec V$.*

This follows from the descending chain condition in $\mathfrak{B}$ (Proposition 1.5).

If there exists a minimal model of Σ/k, then there is only one such model. On the other hand, it follows from Corollary 1.8 that if there does not exist a minimal model of Σ/k and if the class $\mathfrak{B}$ is not empty, then there exist at least two distinct relatively minimal models of Σ/k (distinct = not biregularly equivalent).

2. Exceptional curves of the 1st and 2nd kind. The problem of minimal models for algebraic surfaces requires a preliminary study of exceptional curves on an algebraic surface. This preliminary work has been carried out in Part II of our paper IP and will be briefly reviewed here (without proofs).

DEFINITION 2.1. *An algebraic curve E (reducible or not) on a normal surface F is said to be an exceptional curve of F if there exists a birational transformation $T: F \to F'$ of F onto a surface F' such that E is the* TOTAL *T^{-1}-transform of a* SIMPLE *point of F'*

DEFINITION 2.2. *An exceptional curve E of a normal surface F is said to be* OF THE FIRST KIND *if there exists a birational transformation T satisfying the condition stated in the preceding definition and satisfying the further condition that it be regular at each point of E. In the contrary case E is said to be an exceptional curve* OF THE SECOND KIND.

Given a curve E (on F) and a birational transformation $T: F \to F'$ such that E is the total T^{-1}-transform of a *normal* point P' of F', we shall say, for the sake of brevity, that T *contracts* E into the point P'. This expression, when used in this paper, will therefore always mean a little more than what it actually says; it will mean that not only does every point of E correspond to the point P' in the birational transformation T, but also that no point of F which lies outside the curve E corresponds to P' (and it is also tacitly assumed that P' is a normal point of F').

If a birational transformation $T: F \to F'$ contracts a curve E into a (normal) point P' of F' then we shall say that P' *is a contraction of* E. If, furthermore, T is regular at each point of E then we say that P' *is a regular contraction of* E.

An exceptional curve on a normal surface F is therefore a curve which admits a *simple contraction* (i.e., a contraction P' which is a simple point), and an exceptional curve of the first kind is one which admits a *simple regular contraction*. On the other hand, if E is an exceptional curve of F, *of the second kind*, then *any* birational transformation T of F which contracts E to a *simple* point has necessarily at least one fundamental point *on the curve* E (since F is normal), and thus T "blows up" some points of E into a curve.

PROPOSITION 2.3. *If P' is a simple regular contraction of an exceptional curve E, of the first kind, of a normal surface F, then P' is the only regular contraction of E (in the biregular sense; i.e., if P'' is any regular contraction of E then the local rings $o_{P'}$ and $o_{P''}$ coincide). Similarly, P' is the only simple contraction of E. The point P' is thus uniquely determined by the curve E, being independent of the choice of the birational transformation which contracts E into a simple point. The local ring $o_{P'}$ is the intersection of the local rings o_P, $P \in E$* (see IP, Theorem II.2.3).

It can be shown by examples that an exceptional curve of the first kind has infinitely many non-regular (and therefore necessarily non-simple, by Proposition 2.3) contractions. As to exceptional curves of the second kind, it can be shown that any such curve has infinitely many simple contractions (IP, Corollary II. 5. 15).

Let E be an exceptional curve of the first kind on a *non-singular* surface F, and let $T: F \to F'$ be a birational transformation of F which contracts E to a simple point P'. By Proposition 2.3, the birational transformation T is necessarily regular at each point of E, whence P' is the (uniquely determined) simple regular contraction of E. Nothing in our definition of exceptional curves of the first kind specifies the behavior of T at points of F which are not on E. Thus T may have fundamental points outside of E; T may also contract into points other curves of F (such curves will necessarily have no point in common with E, since E is the *total* T^{-1}-transform of P' and since T is regular, hence single-valued, at each point of E; and the contractions of these curves will necessarily all be different from P'). However, it can be shown that there exists another birational transformation of F which contracts E to P' *and which is biregular on* $F - E$ (see IP, Theorem II. 2. 4, where a more general result is proved). We can formulate this result in the following fashion:

We note first that $F - E + P'$ is easily seen to be an abstract surface in the sense of Weil. Then the above result can be stated as follows:

PROPOSITION 2.4. *If E is an exceptional curve, of the first kind, on a non-singular surface, and if P' is the (uniquely determined) simple (whence also regular) contraction of E, then the abstract surface $F - E + P'$ is a projective surface.*

By repeated applications of Proposition 2.4 we find the following generalization:

PROPOSITION 2.5. *If $E_1, E_2, \cdots, E_s$ are exceptional curves, of the first kind, on a non-singular surface F, such that no two of the curves E_i have points in common, and if P'_i is the simple contraction of E_i, then the abstract surface $F - \sum E_i + \sum P'_i$ is projective.*

The simplest exceptional curve of the first kind which has a preassigned contraction P', where P' is a simple point of a surface F', is the total transform of P' under a locally quadratic transformation of F' with center P'. The exceptional curve E which is thus obtained is irreducible, rational, non-singular, and its numerical characters $p(E)$ (arithmetic genus) and (E^2)

(virtual degree, or self-intersection number) are given by (see IP, Proposition II. 5. 9)

$$p(E) = 0, \qquad (E^2) = -1. \tag{1}$$

Conversely, every irreducible exceptional curve E of the first kind, on a non-singular surface, is the total quadratic transform of its simple construction P' (IP, Corollary II. 3. 2). A deeper result is the following: if an irreducible curve E on a non-singular surface is such that $p(E) = 0$ and $(E^2) = -1$, then E is an exceptional curve of the first kind (see our Ergebnisse monograph "Algebraic surfaces" [11], pp. 71-72).

In IP we have derived a number of properties of reducible exceptional curves, of the first kind, on a non-singular surface (IP, Section II. 3), but we shall make no use of reducible exceptional curves of the first kind in the present paper.

Given a reducible exceptional curve E, of the first kind, on a non-singular surface F, it is possible to define a divisorial cycle $\mathcal{E}$ on F whose prime components are the irreducible components of E, counted to suitable (positive) multiplicities, in such way that relations similar to (1) be satisfied, i. e., so as to have $p(\mathcal{E}) = 0$ and $(\mathcal{E}^2) = -1$. For the definition and properties of these *exceptional cycles of the first kind* (which were not treated in IP) see our cited monograph [11], pp. 36-40; compare also our forthcoming note "Exceptional cycles of the first kind on a non-singular surface and the descending chain condition in birational classes of non-singular surfaces" in the Mem. Coll. Sci. Univ. Kyoto.

We terminate this section with listing a few properties of exceptional curves of the second kind, on a non-singular surface.

PROPOSITION 2. 6. *If a relatively minimal model carries an exceptional curve of the second kind, then it also carries an irreducible exceptional curve of the second kind.* (IP, Corollary II. 4. 6)

PROPOSITION 2. 7. *If E is an irreducible exception curve of the second kind, on a non-singular surface F, and if $s_1, s_2, \cdots, s_m$ are the multiplicities of its singular points (including its infinitely near singular points; if E is non-singular we set m and all the s_i equal to zero), then there exists a non-negative integer n such that*

$$p(E) = \sum s_i(s_i - 1)/2, \tag{2}$$

$$(E^2) = -1 + \sum s_i^2 + n, \quad n + m > 0; \; n, m \geqq 0. \tag{3}$$

It follows, in particular, that $(E^2) \geqq 0$ and that if E has singularities then $(E^2) > 0$. (IP, Corollary II. 5. 14)

We now prove the converse of Proposition 2.7:

PROPOSITION 2.8. *Let E be an irreducible curve on a non-singular surface F, and let $s_1, s_2, \cdots, s_m$ be the multiplicities of its singular points (including its infinitely near singular points). If there exists an integer $n \geqq 0$ such that $n+m>0$ and such that the numerical characters $p(E)$ and (E^2) of E are given by* (2) *and* (3), *then E is an exceptional curve of the second kind.*

Proof. We have only to show that there exists an *antiregular* birational transformation $T: F \to F'$ of F onto a non-singular surface F' (i.e., a *transformation T such that T^{-1} is regular*) such that the *proper* transform $E' = T[E]$ of E is an exceptional curve of the first kind. For assume that this has already been shown. Then, denoting by P'' the simple contraction of E', the antiregularity of T shows immediately that the total transform of P'' *on* F contains the curve E and no other *curves.* Since the proper transform of P'' must be pure one-dimensional, it follows that E is the proper transform of P'', whence E is an exceptional curve. Since $(E^2) \geqq 0$, by (3), E must be of the second kind.

Now, to obtain T we apply to F successive quadratic transformations whose centers are at the singular points of E and of the successive transforms of E, until we obtain a *non-singular* proper transform $\bar{E}$ of E. From (2) and (3) we obtain easily that $p(\bar{E}) = 0$, $(\bar{E}^2) = -1+n$. We then choose n distinct points of $\bar{E}$ and we apply n quadratic transformations with these points as centers. The proper transform of $\bar{E}$ is then an irreducible curve E' such that $p(E') = 0$ and $(E')^2 = -1$, hence an exceptional curve of the first kind. Q. E. D.

If K is a canonical divisor on a non-singular surface F and Z is any divisorial cycle on F we set

$$\sigma(Z) = -(Z \cdot K). \tag{4}$$

It is known (see IP, II. 5, formula (6)) that

$$\sigma(Z) = (Z^2) - 2p(Z) + 2. \tag{5}$$

It follows from (2) and (3) that if E is an irreducible exceptional curve, of the second kind, on F, then

$$\sigma(E) = 1 + n + \sum s_i \geqq 2. \tag{6}$$

Thus $(E \cdot K) < 0$. On the other hand, we have $(E^2) \geqq 0$ and hence $(E \cdot Z) \geqq 0$ for any divisorial cycle $Z \geqq 0$. Consequently, if i is any integer

$\geqq 1$ there cannot exist a divisorial cycle $\geqq 0$ which is linearly equivalent to iK. In other words, the multicanonical systems $|iK|$, $i = 1, 2, 3, \cdots$, do not exist. We have therefore

PROPOSITION 2.9. *If a non-singular surface F carries an irreducible exceptional curve E of the second kind, then all the plurigenera P_i of F ($i \geqq 1$; $P_1 = p_g$) are zero.*

Remark. What matters in the proof of Proposition 2.9 is not that E is an exceptional curve of the second kind but the fact that E is an irreducible curve such that $(E^2) \geqq 0$ and $\sigma(E) > 0$. Consequently, already the presence of such an irreducible curve E implies the vanishing of all the plurigenera.

3. Preliminary criteria of existence of minimal models and statement of the main results. Let F be a member of our birational class $\mathfrak{B}$, i.e., F is a non-singular projective model of the function geld Σ/k. If F carries an exceptional curve E, of the first kind, and if P' is the simple contraction of E, then, by Proposition 2.4, the class $\mathfrak{B}$ contains the surface $F - E + P'$ which is strictly dominated by F, and hence F is not a relatively minimal model of Σ/k. Conversely, if F is not a relatively minimal model of Σ/k, then exists a surface F' in $\mathfrak{B}$ such that $F' < F \neq F'$, the birational transformation $T: F' \to F$ has then necessarily a fundamental point P' on F', and the total transform $T\{P'\}$ is an exception curve on F, of the first kind (since T^{-1} is regular on F, and P' is a simple point). We have therefore

PROPOSITION 3.1. *A necessary and sufficient condition that a surface F in the birational class $\mathfrak{B}$ be a relatively minimal model of Σ/k is that F be free from exceptional curves of the first kind.*

We now prove

PROPOSITION 3.2. *A necessary and sufficient condition that a non-singular projective model F of Σ/k be a minimal model of Σ/k is that F be free from exceptional curves* (of the first *or* of the second kind).

Proof. The condition is obviously sufficient. Suppose now that F carries an exception curve E. Let $T: F \to F'$ be a birational transformation of F which contracts E into a simple point P'. By the theorem of reduction of singularities there exists a non-singular projective model F'' of Σ/k such that $F' \prec F''$. Since P' is a simple point of F' we can choose the resolving birational transformation $F' \to F''$ of F' in such a way that it be biregular at P'. Then also the birational transformation $F \to F''$ contracts E to the

point P'. Thus $F \not\prec F''$, and since $F'' \in \mathfrak{B}$, F is not a minimal model of Σ/k. This completes the proof.

Let E_1, E_2 be two exception curves, of the first kind, on a non-singular surface F, and let P' and P'' be the simple contractions of E_1 and E_2 respectively. If $E_1 \subset E_2$, then it follows from the last part of Proposition 2.3 that $o_{P'} \supset o_{P''}$. (Note that also the converse is true, for if $o_{P'} \supset o_{P''}$ then every zero-dimensional valuation with center P' has necessarily center P'', and thus the total transform E_1 of P' on F is contained in the total transform E_2 of P''). It follows that the projective surface $F - E_1 + P'$ strictly dominates the projective surface $F - E_2 + P''$ (we assume, of course, that $E_1 \neq E_2$). By the descending chain condition in $\mathfrak{B}$ (Proposition 1.5) it follows therefore that *F cannot carry a strictly ascending chain $E_1 \subset E_2 \subset E_3 \cdots$ of exceptional curves of the first kind.* If, then, we call an exceptional curve E of the first kind *maximal* when E is not a proper subset of any other exceptional curve of the first kind, then we conclude that *any exceptional curve, of the first kind, on a non-singular surface F is contained in at least one maximal exceptional curve of the first kind.*

PROPOSITION 3.3. *A necessary condition that the field Σ/k have a minimal model is that every non-singular projective model F of Σ/k have the following three properties:*

(a) *F carries no exceptional curves of the second kind.*

(b) *F has only a finite number of exceptional curves of the first kind.*

(c) *Any two distinct maximal exceptional curves of the first kind have no points in common.*

A sufficient condition that Σ/k have a minimal model is that there exist a non-singular projective model of Σ/k having properties (a), (b) *and* (c).

Proof. Assume that Σ/k has a minimal model, say F^*. Let F be any non-singular projective model of Σ/k. We first show that F can carry only exceptional curves *of the first kind.* Let E be an exceptional curve on F and let $T: F \to F'$ be a birational transformation of F which contracts E to a simple point P' ($P' \in F'$). Without loss of generality we may assume that F' is a non-singular surface (see proof of Proposition 3.2). Let P be any point of E. Since P and P' are corresponding points under T, there exists a zero-dimensional valuation v of Σ/k having centers P and P' on F and F' respectively. Let P^* be the center of v on F^*. Since F^* is a minimal model we have $F^* \prec F$ and $F^* \prec F'$, whence $o_{P^*} \subset o_P$ and $o_{P^*} \subset o_{P'}$. From these inclusions and from the fact that P, P' are simple centers of one and

the same valuation it follows by a known result (see Zariski [12], Lemma on p. 538, or also IP, proof of Theorem II. 1. 1) that either $o_P \subset o_{P'}$ or $o_{P'} \subset o_P$. The first inclusion is impossible since T^{-1} is not regular at P' (the point P' is fundamental for T^{-1}). Hence $o_{P'} \subset o_P$, and thus T is regular at P. Since P was any point of E, the curve E is exceptional *of the first kind*, by Definition 2.1.

We have thus shown that F has property (a). Now let $P_1^*, P_2^*, \cdots, P_s^*$ be the fundamental points of the birational transformation $f: F^* \to F$ and let $E_i = f\{P_i\}$. The s curves E_i on F are exceptional of the first kind (we assume, of course, that $F \neq F^*$, and that therefore $s \geqq 1$, for otherwise conditions (b) and (c) are trivially satisfied), and *no two of these curves have points in common*; both these assertions follow from the regularity of f. Let E be any exceptional curve on F. We know already that E is of the first kind. Let P' be the simple contraction of E, let $F' = F - E + P'$ and let P^* be *the* point of F^* which corresponds to P' in the birational transformation $F' \to F^*$. Since $o_{P^*} \subset o_{P'}$ and P' is fundamental for the birational transformation $F' \to F$, it follows that *a fortiori* also P^* is fundamental for $f: F^* \to F$. Hence P^* is one of the points P_i^*, say $P^* = P_1^*$. Since $o_{P^*} \subset o_{P'}$, it follows, by a remark made earlier, that the total transform E_1 of P^* on F contains the total transform E of P' on F. We have therefore shown that every exceptional curve E, of the first kind, on F is contained in one of the s exceptional curves E_i. Hence F carries exactly s maximal exceptional curves of the first kind, and this proves (b). Since no two of E_i intersect, the necessity of the condition is established.[3]

Assume now that there exists a non-singular projective model F of Σ/k having properties (a), (b) and (c). Let $E_1, E_2, \cdots, E_s$ be the maximal exceptional curves, of the first kind, on F and let P_i^* be the simple contraction of E_i. Since no two of E_i intersect, the abstract surface $F^* = F - \sum E_i + \sum P_i^*$ is projective (Proposition 2.5). We shall show that F^* is a minimal model of Σ/k.

Let F' be any non-singular projective model of Σ/k and let $f: F' \to F$ and $f^*: F' \to F^*$ be the birational transformations of F' onto F and F^* respectively. Let P' be any point of F'. If f is regular at P' then also f^* is regular at P' since $F^* \prec F$. If f is not regular at P', let $E = f\{P'\}$ be the exceptional curve on F which is the total f-transform of the fundamental

[3] We observe that even if there does not exist a minimal model the following is true for any non-singular projective model F of Σ/k: there does not exist on F an infinite set of exceptional curves of the first kind such that no two curves in the set intersect. This is a straightforward consequence of the descending chain condition in the class $\mathfrak{B}$.

point P' of f. The exceptional curve E is of the first kind, since F has property (a), and E is contained in one of the s maximal exceptional curves E_i, say $E \subset E_1$. Since P' and $P_1{}^*$ are the simple contractions of E and E_1 respectively, it follows from $E \subset E_1$ that $o_{P'} \supset o_{P_1{}^*}$. Thus also in this case the birational transformation $f^*: F' \to F^*$ is regular at the point P'. Since P' is any point of F', it follows that $F' \succ F^*$. Since F' is any non-singular projective model of Σ/k the assertion that F^* is a minimal model of Σ/k is established. This completes the proof of the proposition.

Our principal object in this paper is the characterization of those function fields Σ/k of transcendence degree 2 which have no minimal model. This characterization is given by the following theorem (due to the Italian geometers Castelnuovo and Enriques in the classical case):

FUNDAMENTAL THEOREM A. *If the field Σ/k does not possess a minimal model then Σ is the field of rational functions of a ruled surface (or equivalently: Σ is a simple transcendental extension $\Omega(t)$ of a field Ω/k of algebraic functions of one variable).*

The converse of the fundamental theorem A is also true, but is trivial. For, if $\Sigma = \Omega(t)$ and if C denotes a non-singular curve such that $k(C) = \Omega$, then the direct product $C \times L$, where L is a projective line, is a non-singular projective model of Σ/k (the Segre variety of the pairs of points of C and L). The curves E on $C \times L$, given by $Q \times L$ $(Q \in C)$, are irreducible, non-singular, and we have $p(E) = 0$, $(E^2) = 0$. Hence each E is an exceptional curve of the second kind (Proposition 2.8), and therefore $\Omega(t)$ has no minimal model (Proposition 3.3).

We shall prove the fundamental theorem A in the following stronger form:

FUNDAMENTAL THEOREM B. *If a non-singular surface F carries an irreducible exceptional curve E of the second kind, then F is birationally equivalent to a ruled surface, and if, furthermore, $(E^2) > 0$ (in particular, if E has singular points; see Proposition 2.7) then F is a rational surface.*

The remainder of this paper is devoted to the proof of this theorem. In regard to this theorem we point out at this stage that if the theorem is true under the additional assumption that F is a relatively minimal model then it is true in general. For assume that the theorem has already been proved under the assumption that F is a relatively minimal model and let F any non-singular surface satisfying the conditions of the fundamental theorem B. By the descending chain condition in the birational class $\mathfrak{B}$ of F, there

exists a relatively minimal model F^* in B such that $F^* \prec F$. Let $T: F \to F^*$ be the (regular) birational transformation of F onto F^*. We assert that E is not an exceptional curve of T, i.e., that $T\{E\}$ is not a point. For assume the contrary, and let $T\{E\}$ be a point P^* of F^*. Let P' be a simple contraction of E and let F' be a non-singular surface which carries the point P'. From the regularity of T and from the fact that E is the total transform of P' it follows that P^* is the only point of F^* which corresponds to the point P' in the birational transformation $F' \to F^*$. Hence $\mathfrak{o}_{P^*} \subset \mathfrak{o}_{P'}$. Since we have also $\mathfrak{o}_{P^*} \subset \mathfrak{o}_P$ for any point P of E, the argument developed in the first part of the proof of Proposition 3.3 shows that $\mathfrak{o}_{P'} \subset \mathfrak{o}_P$. This being true for every point P of E, P' is a simple *regular* contraction of E, in contradiction with the fact that E is an exceptional curve of the *second* kind. [What we have proved here is that no irreducible component of an exceptional curve of the first kind (in the present case—of $T^{-1}\{P^*\}$) can be an exceptional curve of the second kind.]

Let then $T\{E\} = E^*$, where E^* is an irreducible curve on F^*. From the regularity of T and from the irreducibility of E it follows that also E^* is an exceptional curve (every simple contraction of E is also a contraction of E^*). The regularity of T implies that $(E^{*2}) \geqq (E^2)$ (IP, II 5, formula (12)). Since $(E^2) \geqq 0$ it follows that also $(E^{*2}) \geqq 0$, and consequently E^* is of the second kind. If, furthermore, $(E^2) > 0$ then also $(E^{*2}) > 0$, F^* is rational, and so is F.

The proof of the fundamental theorem will be divided in several cases, according as (E^2) is zero or positive, and in the latter case according to the value of (K^2), where K is a canonical divisor. In view of what was just said above, we may assume, in any given case, that F is a relatively minimal model. In some cases this assumption will play a crucial role in the proof, while in other cases that assumption is not required and will in fact not be made.

4. Case $(E^2) = 0$. *Beginning with this section, and throughout the rest of the paper, we assume that the field Σ/k has no minimal model.* We fix a relatively minimal model F of Σ/k. The surface F must then carry an *irreducible* exceptional curve E, of the second kind (Propositions 3.1, 3.2 and 2.6). In this section we assume that F carries an irreducible exceptional curve E, of the second kind, such that $(E^2) = 0$. Then E is non-singular and $p(E) = 0$ (Proposition 2.7), whence E is a non-singular rational curve.

We proceed to prove that *any non-singular surface F which carries an irreducible non-singular rational curve E such that* $(E^2) \geqq 0$ (such a curve

is in fact exceptional of the second kind; see Proposition 2.8) *is birationally equivalent to a ruled surface.*

We first consider the case $(E^2) = 0$. By formula (5) of Section 2 we have $\sigma(E) = 2$ and consequently $\sigma(nE) = 2n$ for any integer n.

If Z is any divisorial cycle on F then by the Riemann-Roch theorem

$$\dim |Z| \geqq \tfrac{1}{2}[\sigma(Z) + (Z^2)] + p_a - i(Z), \tag{1}$$

where $i(Z) = 1 + \dim |K - Z|$ and p_a is the arithmetic genus of F. If $Z \geqq 0$ then $i(Z) = 0$ since the canonical system dos not exist on our surface F (see *Remark* following Proposition 2.9). Applying (1) to the cycle $Z = nE$ we find

$$\dim |nE| \geqq n + p_a, \qquad n \geqq 0. \tag{2}$$

For any integer $n \geqq 1$ we have only the following two possibilities: (a) either E is a fixed component of $|nE|$, and in that case $\dim |nE| = \dim |n-1)E$, or (b) E is not a fixed component of $|nE|$, and in that case $\dim |nE| = 1 + \dim |(n-1)E|$, since $(E^2) = 0$ and since therefore the trace of $|nE|$ on E consists only of the zero-cycle on E. Hence in both cases we have $0 \leqq \dim |nE| - \dim |(n-1)E| \leqq 1$, and from this it follows, in view of (2) that *for large* n we have

$$\dim |nE| = n + c, \tag{3}$$

where c is a constant, and

$$\dim |nE| = 1 + \dim |(n-1)E|. \tag{4}$$

Since E is irreducible, it follows from (4) that for large n the system $|nE|$ has no fixed components. On the other hand, if $n > 1$ the system $|nE|$ must be reducible since any two cycles in $|nE|$ which pass through a given point of F must have a common component (the intersection number $(nE \cdot nE)$ being zero)). Therefore, for large n, the system $|nE|$ is composite with a pencil, or else all the members of $|nE|$ are the p-folds of other cycles, where p is the characteristic of k (Theorem of Bertini; see IP, Theorem I.6.3). The second alternative is to be excluded, for if n is large, then also $|(n-1)E|$ has no fixed components and thus $|nE|$ contains a cycle of the form $E + D$, where $D \in |(n-1)E|$ and E is not a component of D, so that E is a simple component of that cycle. Therefore $|nE|$ is composite with a pencil (not neecssarily linear) and thus it is also composite with a (uniquely determined) irreducible pencil, say L_n. Since $nE \in |nE|$, some multiple of E, say s_nE, belongs to L_n. Then every cycle in $|nE|$ which contains E as

component must contain E to multiplicity $\geqq s_n$. Since we have just seen that if n is large, then there exist cycles in $|nE|$ which contain E as a simple component, we conclude that $s_n = 1$, i.e., E itself is a member of the pencil L_n. Since E is irreducible and $(E^2) = 0$, E cannot belong to two distinct pencils. Hence L_n does not depend on n, and the systems $|nE|$, n large, are composite with one and the same irreducible pencil $L = L_n$.

We have thus found a pencil L on F one member of which, namely E, is a prime cycle whose arithmetic genus is zero. But then also the general cycle of L, over k, has arithmetic genus zero. By the theorem of Noether-Enriques (or as a consequence of the purely algebraic theorem of Tsen [8]),[4] it follows that F is birationally equivalent to a ruled surface.

The preceding proof is applicable also to the case $(E^2) > 0$ provided E is *non-singular*, and in that case the conclusion is that *the surface F is rational.* In fact, if $(E^2) = s > 0$, then after s quadratic transformations with center on E and the successive transforms of E we obtain a non-singular surface F' on which the proper transform of E is an irreducible, non-singular, rational surve E' such that $(E'^2) = 0$. By the preceding proof, E' is a member of a pencil L'. The corresponding pencil L on F has s base points ($s > 0$) and is therefore a linear pencil (Schilling-Zariski [6]). Therefore also L' is a linear pencil, and F is a rational surface (the field Ω of footnote 4 is now a simple transcendental extension of k).

5. Case $(E) > 0$. The vanishing of $p_a(F)$. We now assume that our relatively minimal model F carries an irreducible exceptional curve E, of the second kind, such that the self-intersection number (E^2) of E *is positive.* Our object in the rest of the paper is to prove that F is a rational surfaces. In this section we shall carry out a first step of the proof, namely, we shall show that the arithmetic genus p_a *of F is zero.*

We shall need the following lemma:

LEMMA 5.1. *Let $T: G \to G'$ be a rational regular transformation of a non-singular surface G onto a (not necessarily non-singular) variety G' of*

[4] The pencil L determines a subfield Ω of Σ/k, of transcendence degree 1. The fact that the general member of L/k has arithmetic genus zero signifies that if Ω is taken as ground field then Σ—a field of algebraic functions of *one* variable over Ω—has genus zero. Thus Σ/Ω has divisors of degree 2 (the anticanonical divisors of Σ/Ω), and the Riemann-Roch theorem yields a projective model C of Σ/Ω which is a conic, defined over Ω. Since k is algebraically closed, the theorem of Tsen guarantees the existence of a rational point of C, over Ω, and consequently C is birationally equivalent, over Ω, to a projective line. Hence Σ is a purely transcendental extension of Ω.

positive dimension (whence G' is either a surface or a curve). If E is an irreducible curve on F such that $T\{E\}$ is a point then $(E^2) \leqq 0$.

Proof. Let L be the linear system on G, free from fixed components, which corresponds to the system of hyperplane sections of G' (see IP, I. 3 and I. 5). Since T is regular, L has no base points. Since E is exceptional for T, the linear system $Tr_E L$ has dimension zero, i. e., this linear system consists only of one divisor (on E). This divisor must be the null divisor, for L has no base points. Hence $(D \cdot L) = 0$ for any cycle D in L. Furthermore, if $\dim L = m$ (where necessarily $m \geqq 1$), then L contains a linear subsystem L_1, of dimension $m-1$, having E as fixed component. Fix a cycle $sE + \Delta$ in L_1, where $\Delta \geqq 0$, $s \geqq 1$ and where we assume that E is not a component of Δ. Then $(\Delta \cdot E) \geqq 0$, and since $(sE + \Delta \cdot E) = 0$, it follows that $(E^2) \leqq 0$, as asserted.[5]

We now go back to our relatively minimal model F.

PROPOSITION 5. 2. *The irregularity q of F is zero ($q =$ dimension of the Albanese variety of F).*

Proof. Let A be the Albanese variety of F, let f be the canonical mapping of F into A and let $F' = f(F)$. Since F is non-singular it follows, by a well-known property of Abelian varieties (A. Weil, [9], Th. 6, p. 27) that the rational transformation f of F onto F' is regular. Since E is a rational curve and since A carries no rational curves, *the transform $f(E)$ of E must be a point* (for otherwise $f(E)$ would be a *curve*, which—by the theorem of Lüroth—would then be rational). Since $(E^2) > 0$, it follows from the preceding lemma that F' cannot have positive dimension. Hence F' is a point, and since F' generates A, also A is a point. Q. E. D.

In the classical case, Severi has proved, by an essentially algebraic method, the completeness of the characteristic series of any ample complete continuous system of curves on a surface F whose geometric genus p_g is zero (see Severi [7], Anhang F; see also our monograph [11], pp. 84-86, where the proof and references may be found). In other words, Severi has proved, in the classical case, the *equality* $q = p_g - p_a$ by algebraic methods, under the assumption $p_g = 0$. This equality, under the same assumption, has now been established also in the abstract case, and independent proofs have been given by Akizuki-Matsumura [1], Nakai [4], [5] and Matsumura-Nagata

[5] Note that if F' is a *surface* then one can conclude that $(E^2) < 0$. For in that case we have $m \geqq 2$, $\dim L_1 > 0$, and thus we can find a cycle $sE + \Delta$ in L_1 such that $\Delta > 0$. By the connectedness theorem (Zariski [13], Theorem 14, p. 77) it then follows that $(\Delta \cdot E) > 0$, whence $(E^2) < 0$.

[3]. Since we know that for our surface F we have $p_g = 0$ (Proposition 2.9), it follows from Proposition 5.2 that $p_a = 0$. This completes the preliminary step of our proof of rationality of F.

In view of the vanishing of p_a, the Riemann-Roch inequality on the surface F is now as follows:

$$\dim |Z| \geqq \tfrac{1}{2}[\sigma(Z) + (Z^2)] - i(Z), \tag{1}$$

where $i(Z) = 1 + \dim |K - Z|$ and $\sigma(Z) = -(Z \cdot K)$ (Z, a divisorial cycle; K, a canonical cycle). Since $p_g = 0$ we have $i(Z) = 0$ if $Z \geqq 0$.

We denote by Z' any cycle which is linearly equivalent to $Z + K$, whence $|Z'|$ is the so-called *adjoint system of* $|Z|$. We set

$$(K^2) = \nu. \tag{2}$$

Then

$$\sigma(Z') = \sigma(Z) - \nu, \quad (Z'^2) = (Z^2) + \nu - 2\sigma(Z), \quad i(Z') = 1 + \dim |-Z|.$$

Hence, applying (1) to Z' instead of to Z, and using the equality $\sigma(Z) = (Z^2) - 2p(Z) + 2$, we find

$$\dim |Z'| \geqq p(Z) - \dim |-Z| - 2, \tag{3}$$

and if $|Z|$ *contains positive cycles then*

$$\dim |Z'| \geqq p(Z) - 1, \qquad Z > 0. \tag{4}$$

6. The case $(E^2) > 0$, $(K^2) \leqq 0$. In this section we prove the fundamental theorem B under the assumptions that $(E^2) > 0$, $(K^2) \leqq 0$ and that F is a relatively minimal model.

We assert that *the linear system* $|E + nK|$ (*the n-th adjoint system of* $|E|$) *does not exist if n is large.* In fact, we have $\sigma(E) \geqq 2$ [see (6), § 2], i.e., $(E \cdot K) \leqq -2$, whence $(E + nK \cdot E) < 0$ if n is large. Hence, if the system $|E + nK|$ exists for large n, then that system would have the curve E (which is *irreducible*) as fixed component, in contradiction with the non-existence of $|nK|$ (Proposition 2.9).

Let m be the smallest integer ($m \geqq 0$) such that $|E + mK|$ exists and $|E + (m+1)K|$ does not exist. Since $(mK \cdot mK) = m^2(K^2) \leqq 0$ while $(E^2) > 0$, we have $E + mK \not\equiv 0$. Hence the system $|E + mK|$ consists of positive cycles. Let Z be a positive cycle in that system, and let D be any prime component of Z. Since $|Z'|$ does not exist, *a fortiori* $|D'|$ does not exist. Hence by (4), § 5, we must have $p(D) = 0$. On the other hand, since $\sigma(Z) = \sigma(E) - m(K^2) > 0$ we must have $\sigma(D) > 0$ for at least one prime

component D of Z. For such a D we have therefore $(D^2)+2>0$ (since $p(D)=0$). The case $(D^2)=-1$ is excluded since F carries no exceptional curves of the first kind. Hence $(D^2)\geqq 0$, and thus F is birationally equivalent to a ruled surface (§ 4). Since $p_a(F)=0$, F is rational.

7. Some general properties of surfaces with $p_a=P_2=0$ and $(K^2)>0$. In this section we consider a non-singular surface F for which the following conditions are satisfied: $p_a=P_2=0$ and $(K^2)>0$. We shall derive some general properties of such a surface and we shall use these properties in the rest of the paper.

We set $(K^2)=\nu$ and we denote by K_a any *anticanonical* cycle, i.e., any cycle of the form $-K$. We note first of all that

$$p(K_a)=1. \tag{1}$$

This equality holds on *any* non-singular surface, for we have $\sigma(K_a)=-(K_a\cdot K)=(K_a^2)$, and (1) follows from (5), 2. From $\sigma(K_a)=\nu$ we find by (1), § 5:

$$\dim|K_a|\geqq\nu\geqq 1, \tag{2}$$

since $i(K_a)=\dim|2K|+1=0$. Hence the anticanonical system $|K_a|$ exists and has positive dimension. *A fortiori*, all the multi-anticanonical systems $|nK_a|$ $(n>0)$ exist and have positive dimension. Therefore none of the multicanonical systems $|nK|$ $(n>0)$ exists, and thus all the plurigenera of F vanish.

Let Z be any divisorial cycle on F and let n be an integer >0. Assume that the system $|Z+nK|$ exists. Then if we fix a *positive* cycle K_a in $|K_a|$, the linear system $|Z|$ will contain a cycle which is $\geqq nK_a$. Since, for a given Z, this is not possible if n is large, it follows that *the system* $|Z+nK|$ *does not eixst if* n *is large.* Thus the process of successive adjunctions, applied to a cycle Z, always terminates after a finite number of steps.

LEMMA 7.1. *If D is an irreducible curve on F and $|D'|$ does not exist, then $p(D)=0$ (and hence D is a rational non-singular curve).*

Proof. By (4), § 5, the non-existence of $|D'|$ implies $p(D)\leqq 0$, and hence $p(D)=0$ since D is an irreducible curve.

LEMMA 7.2. *Let K_a be a positive anticanonical cycle. If K_a is not a prime cycle, then every prime component of K_a has arithmetic genus zero.*

Proof. Let D *be a prime component of* K_a *and let* $K_a=D+\Delta$, $\Delta>0$

[3]. Since we know that for our surface F we have $p_g = 0$ (Proposition 2.9), it follows from Proposition 5.2 that $p_a = 0$. This completes the preliminary step of our proof of rationality of F.

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component D of Z. For such a D we have therefore $(D^2)+2>0$ (since $p(D)=0$). The case $(D^2)=-1$ is excluded since F carries no exceptional curves of the first kind. Hence $(D^2)\geqq 0$, and thus F is birationally equivalent to a ruled surface (§ 4). Since $p_a(F)=0$, F is rational.

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LEMMA 7.1. *If D is an irreducible curve on F and $|D'|$ does not exist, then $p(D)=0$ (and hence D is a rational non-singular curve).*

Proof. By (4), § 5, the non-existence of $|D'|$ implies $p(D)\leqq 0$, and hence $p(D)=0$ since D is an irreducible curve.

LEMMA 7.2. *Let K_a be a positive anticanonical cycle. If K_a is not a prime cycle, then every prime component of K_a has arithmetic genus zero.*

Proof. Let D *be a prime component of* K_a *and let* $K_a=D+\Delta$, $\Delta>0$

We have $|D'| = |K_a - \Delta + K| = |-\Delta|$. Hence $|D'|$ does not exist, and we can apply the preceding lemma.

PROPOSITION 7.3. *If $|K_a|$ contains a reducible linear subsystem L, of positive dimension, then F is a rational surface.*

Proof. The system L may have fixed components. Let B be the maximal fixed component of L and let L_1 be the system obtained from L by deleting the fixed cycle B (so that L_1 is a linear system of positive dimension and free from fixed components). Let the cycle A be defined as follows: 1) If L_1 is irreducible then A is a general cycle of L_1/k. 2) if L_1 is reducible, then L_1 is either composite with an irreducible pencil or the cycles of L_1 are all of the form p^eC, where C varies in an irreducible linear system M of positive dimension; in the first case, A shall be a general cycle of that pencil, in the second case it shall be a general cycle of the linear system M/k. In all cases, A is a prime cycle and is a component of a positive anticanonical cycle which is not prime. Therefore, by Lemma 7.2, we have $p(A) = 0$. We have thus on F either an irreducible linear system of positive dimension, whose general cycle A has arithmetic genus zero, in which case F is a rational surface, by the theorem of Noether-Enriques; or a pencil whose general member A has arithmetic genus zero. In the latter case, F must be birationally equivalent to a ruled surface, again by the theorem of NoetherEnriques. But since $p_a = 0$, this ruled surface is of genus zero, and so again F is rational.

8. Proof of Castelnuovo's criterion of rationality in the case $(K^2) \geqq 3$. We again assume that $p_a = P_2 = 0$ and we consider first the case $(K^2) = \nu \geqq 4$. We fix a rational point P of F, over k, and we denote by L the linear cystem of all cycles in $|K_a|$ which have at P a singular point (hence at least a double point). We have $\dim L \geqq \dim |K_a| - 3$, whence, by (2), § 7: $\dim L \geqq \nu - 3 \geqq 1$. If L is reducible, then F is rational (Proposition 7.3). Assume that L is irreducible. Since $p(K_a) = 1$ it follows that the general member of L/k is a rational curve (which, however, as a cycle has arithmetic genus 1). Let $T: F \to \bar{F}$ be a quadratic transformation of F, with center P. For every cycle D in L we have $T(D) = \bar{D} + 2\bar{E}$, where $\bar{E} = T\{P\}$ is the irreducible exceptional curve of the first kind created by T and where $\bar{D}$ is a positive cycle. As D varies in L, $\bar{D}$ varies in a linear system $\bar{L}$, of positive dimension, and since $p(D) = 1$ it follows at once that $p(\bar{D}) = 0$. Thus $\bar{F}$ carries a linear system (of positive

dimension) of cycles having arithmetic genus zero, the general cycle of the system being prime. Hence $\bar{F}$, and therefore also F, is rational.

We consider now the case $\nu = 3$. By Proposition 7.3 we may assume that $|K_a|$ is irreducible (since $\dim |K_a| \geqq 3$). If r denotes the dimension of $|K_a|$, the system $|K_a|$ defines a rational transformation $T: F \to F'$ of F onto an algebraic variety F' of positive dimension $\leqq 2$, immersed in a projective space of dimension r, whose system of hyperplane sections corresponds to $|K_a|$. Since $|K_a|$ is not composite with a pencil, F' is a surface (not a curve). Since the degree of $|K_a|$ is the prime integer 3 and the dimension is $\geqq 3$, the system $|K_a|$ cannot be composite with any unramified involution. Hence, the function field $\Sigma = k(F)$ either coincides with the function field $k(F')$ or is a purely inseparable extension of $k(F')$, of degree 3. In the latter case, F' would be a surface of *order* 1, i.e., a plane, which is impossible, since F' does not lie in a projective space of dimension $< r$, and $r \geqq 3$. Hence $k(F) = k(F')$, and F' is *a surface of* order $\geqq 3$, *birationally equivalent to* F (it could be a surface of order < 3, for *a priori* $|K_a|$ may have base points; see, however, Lemma 9.1 in the next section). If F' is a surface of order < 3, or if F' is a cubic surface and $r \geqq 4$, then F' is rational [if $r \geqq 4$ then a central projection of F' transforms F' birationally onto a quadratic surface; actually, it can be shown that $r = 3$ (see Lemma 9.1 in the next section)]. If F' is a cubic surface in S_3 ($r = 3$) then F' cannot be an elliptic cubic cone since $p_a(F) = 0$ and since F and F' are birationally equivalent. Hence F' is necessarily a rational surface. This completes the proof of Castelnuovo's criterion in the case $\nu \geqq 3$.

9. Proof of the fundamental theorem B in the case $(K^2) = 1$. In this case we shall use the following result:

PROPOSITION 9.1. *Let F be a relatively minimal model of a field Σ/k of algebraic functions of two variables and assume that $p_a = P_2 = 0$ and that $(K^2) = \nu > 0$. If there exists on F a divisorial cycle which is not linearly equivalent to an integral multiple of K, F is rational.*

Proof. We first show that *F carries at least one irreducible curve having arithmetic genus zero.* Assume the contrary. Since there exist on F divisorial cycles which are not linearly equivalent to an integral multiple of K, there also exist positive divisorial cycles having this property. Fix one such positive cycle Z. Let D be a prime component of Z. Since, by assumption, $p(D) \geqq 1$, the adjoint system $|D'|$ exists (Lemma 7.1). *A fortiori*, also the adjoint system $|Z'|$ $(= |D' + Z - D|)$ exists. The system $|Z'|$ must

contain positive cycles since $Z \not\equiv -K$. Obviously no cycle in $|Z'|$ can be linearly equivalent to an integral multiple of K. Therefore, replacing Z by a cycle belonging to $|Z'|$ and applying the same argument we deduce that also the second adjoint system $|Z''|$ $(=|Z'+K|)$ exists. Thus all the successive adjoint system $|Z+nK|$, $n=1,2,\cdots$, exist, and this is in contradiction with what has been shown in the beginning of Section 7.

The proposition will be established if we show that either F is rational or F carries an irreducible curve E such that $p(E)=0$ *and* $(E^2)\geqq 0$. For in the latter case F is birationally equivalent to a ruled surface (Section 4) and is therefore again rational, since $p_a=0$.

Assume that for *any* irreducible curve E on F such that $p(E)=0$ we have necessarily $(E^2)<0$, and therefore $(E^2)\leqq -2$ (since $F-a$ relatively minimal model—carries no exceptional curves of the first kind (Proposition 3.1)). We fix an irreducible curve E on F such that $p(E)=0$. Since $(E^2)\leqq -2$, we have $\sigma(E)=(E^2)-2p(E)+2\leqq 0$, i.e., $(E\cdot K_a)\leqq 0$. If $(E\cdot K_a)<0$, then E is a fixed component of $|K_a|$, thus $|K_a|$ is reducible and F is rational (Proposition 7.3). We are trus led to the only remaining possibility: if E is any irreducible curve on F such that $p(E)=0$ then $(E^2)=-2$, whence $\sigma(E)=(K_a\cdot E)=0$. If we fix such a curve E, it is a fundamental curve of $|K_a|$ (in view of $(K_a\cdot E)=0$). Thus there exist a cycle $\bar{K}_a$ in $|K_a|$ which has E as component. For *every* irreducible component $\bar{E}$ of $\bar{K}_a$ we have $p(\bar{E})=0$ (Lemma 7.2), whence $\sigma(\bar{E})=0$. This is in contradiction with $\sigma(\bar{K}_a)=\nu>0$ and with the additive property of the numerical character σ. This completes the proof of the proposition.

Using the above proposition we shall now prove the fundamental theorem B in the case $(K^2)=1$ assuming—as we may—that F is a relatively minimal model. It will be sufficient to show that our (irreducible) exceptional curve E of the second kind cannot be linearly equivalent to an integral multiple of K. Assume the contrary: $E\equiv qK_a$, where q is necessarily a positive integer since $|K_a|$ contains positive cycles. Then $(E^2)=q^2$ and $\sigma(E)=q$. Using formulae (2) and (3) of Section 2 (Prop. 2.7) we find

$$q^2=-1+\sum_{i=1}^{h}s_i^2, \qquad q=1+\sum_{i=1}^{h}s_i,$$

where $h>0$, $s_i>0$ $(i=1,2,\cdots,h)$ and where, in the notations of Proposition 2.7, we have set $h=n+m$ and $s_{m+1}=s_{m+2}=\cdots=s_h=1$. Hence $(1+\sum_{i=1}^{h}s_i)^2=\sum_{i=1}^{h}s_i^2-1$, or $1+\sum_{i<j}s_is_j+\sum s_i=0$, which is impossible.

10. Proof of Castelnuovo's criterion of rationality in the case $(K^2)=2$. We shall need the following lemma which deals with the general case of a non-singular surface F for which $p_a=P_2=0$ and ν $[=(K^2)]$ is an arbitrary positive integer.

LEMMA 10.1. *Let F be a non-singular surface such that $p_a=P_2=0$ and $\nu=(K^2)>0$. Then either F is rational or the following is true: the linear system $|K_a|$ is irreducible, dim $|K_a|=\nu$ and, furthermore, if $\nu\geqq 2$, then $|K_a|$ has no base points and no fundamental curves.*

Proof. If $|K_a|$ is reducible, then F is rational (Proposition 7.3). Assume that $|K_a|$ is irreducible. We show that the assumption dim $|K_a|>\nu$ leads to a contradiction, and that consequently dim $|K_a|=\nu$, by (2), § 7. If dim $|K_a|>\nu$, then the characteristic series of $|K_a|$ on a general member K_a^0 of $|K_a|/k$ is a linear series of degree ν and of dimension $\geqq\nu$ (hence necessarily of dimension ν, so that dim $|K_a|=\nu+1$). This implies that K_a^0 must have a singular point Q. But then the cycles of $|K_a|$ which pass through Q would cut out on K_a^0 a linear series of degree $\leqq\nu-2$ and of dimension $\geqq\nu-1$, which is impossible.

Assume now that $\nu\geqq 2$ (and that $|K_a|$ is irreducible). If $|K_a|$ has base points, then the characteristic series of $|K_a|$ on K_a^0, after the fixed cycle at the base points has been deleted, would be a linear series L of degree $<\nu$ and of dimension $\nu-1$. Thus again K_a^0 would be of effective genus zero and would have to have a singular point Q. The point Q cannot be a base point, for in the contrary case the above series L would be actually of degree $\leqq\nu-2$ (and of dimension $\nu-1$), which is impossible. This being so, the cycles of $|K_a|$ which pass through Q would cut out on K_a^0, outside the fixed cycle at the base points *and* at Q, a series L_1 of degree $\leqq\nu-3$ and of dimension $\nu-2$, which again is impossible (note that L_1 is defined, since $\nu\geqq 2$ and since therefore dim $|K_a|\geqq 2$, so that there exist cycles in $|K_a|$, *different from* K_a^0, which pass through Q).

Assume now that $|K_a|$ has fundamental curves (always under the assumption that $\nu\geqq 2$) and that $|K_a|$ is irreducible, and let Γ be an irreducible fundamental curve of $|K_a|$. Then there exist cycles D in $|K_a|$ which contain Γ as component, and these cycles form a linear system of dimension $\nu-1>0$. Therefore F is rational (Proposition 7.3). This completes the proof of the lemma.

The following is a generalization of the preceding lemma:

LEMMA 10.2. *Under the same assumptions as those of Lemma 8.1, either F is rational, or for any integer $n>1$ the following is true: the system*

$|nK_a|$ is irreducible, has dimension $\frac{1}{2}[\nu n(n+1)]$, is free from base points, and furthermore it is also free from fundamental curves if $\nu \geqq 2$ or if F is a relatively minimal model.

Proof. We may assume that $|K_a|$ is irreducible (otherwise F is rational). Applying the Riemann-Roch inequality (1), Section 5, to the cycle $Z = nK_a$, we find

$$\dim |nK_a| \geqq \tfrac{1}{2}[\nu n(n+1)]. \tag{1}$$

From the irreducibility of $|K_a|$ and from the fact that $\dim |K_a| > 0$, it follows that $|nK_a|$ can mave no fixed component. If $\nu > 1$ that $\dim |K_a| \geqq 2$ and hence $|nK_a|$ cannot be composite with a pencil. If $\nu = 1$ then $|K_a|$ is a pencil, and this could be the only irreducible pencil with which $|nK_a|$ might be composite. But were $|nK_a|$ composite with the pencil $|K_a|$, then we would have $\dim |nK_a| \leqq n$, in contradiction with (1) which yields $\dim |nK_a| \geqq \frac{1}{2}[n(n+1)] > n$ if $n > 1$. Hence $|nK_a|$ is not composite with a pencil. It is also clear that not all the cycles in $|nK_a|$ are p-folds of other cycles, for $|nK_a|$ contains sums of n distinct primes cycles belonging to $|K_a|$. We therefore conclude that $|nK_a|$ is irreducible.

We shall now prove, by induction on n, that $\dim |nK_a| = \nu n(n+1)/2$. Since this equality is valid for $n = 1$ (by the above Lemma 10.1), we assume $n \geqq 2$. We denote by r the dimension of the linear series $g = Tr_C |nK_a|$, where C is a general member of $|K_a|/k$. Then g is a series of degree $n\nu$, and we have $\dim |nK_a| = 1 + r + \dim |(n-1)K_a|$, whence, by induction hypothesis, $\dim |nK_a| = 1 + r + \frac{1}{2}[\nu n(n-1)]$. By (1) we must have $r \geqq \nu n - 1$. Were $\dim |nK_a| > \frac{1}{2}[\nu n(n+1)]$, we would have $r \geqq \nu n$, and the linear series g would have degree νn *and* dimension νn. That would imply that C has effective genus zero. Since $p(C) = 1$ [see (1), § 7], C would have to have a double point Q. The linear series cut out on C by the cycles of $|nK_a|$ passing through Q would then have degree $\leqq \nu n - 2$ and dimension $\geqq \nu n - 1$, which is impossible.[6] This proves the equality $\dim |nK_a| = \nu n(n+1)/2$.

If $\nu \geqq 2$, then $|K_a|$ has no base points or fundamental curves, and this implies that also $|nK_a|$ has this same property. In the case $\nu = 1$ we use an indirect argument. Assume that $|nK_a|$ has a base point A $(n \geqq 2)$.

[6] Note that the cycles in $|nK_a|$ which pass through the point Q form a linear system of dimension $\geqq \nu n(n+1)/2 - 1$, while the cycles in $|nK_a|$ which contain C as component form a linear system of dimension $\nu n(n-1)/2$. If $n \geqq 2$ the former dimension is greater than the latter, and thus there exist cycles in $|nK_a|$ which pass through Q and do not contain C as component. Therefore the linear series in question is defined.

Then A is also a base point of $|K_a|$, $A \in C$, and the series $g = Tr_C |nK_a|$, outside of a fixed cycle at A, has degree $\leqq n-1$. Since the dimension r of g is $n-1$, C is of effective genus zero and must have a double point Q. The point Q is different from A, for otherwise the above series g, outside a fixed cycle at A, would have degree $\leqq n-2$ and dimension $n-1$, which is impossible. The cycles of $|nK_a|$ passing through Q would cut out on C a series of degree $\leqq n-3$ and of dimension $n-2$, which is impossible.

For any irreducible fundamental curve E of $|nK_a|$, $n>1$, we must have $(nK_a \cdot E) = 0$, since $|nK_a|$ has no base points. This implies that E is not linearly equivalent to any integral multiple of K_a. Hence if F is relatively minimal model, F is rational (Proposition 9.1).

This completes the proof of the lemma. Note that Proposition 9.1 has been used only at the end of the proof, and hence only in the case $\nu = 1$, since in the case $\nu \geqq 2$ the non-existence of fundamental curves of $|nK_a|$ has already been established in the earlier part of the proof.

After these preliminaries we begin to treat the case $\nu = 2$. We assume that we have a surface F which is a relatively minimal model of Σ/k and that we have $p_a = P_2 = 0$, $(K^2) = 2$. We also assume that $|K_a|$ is irreducible (Proposition 7.3). Then $|K_a|$ has no base points and has dimension 2 (see proof of Lemma 10.1). We may also assume that $|K_a|$ has no fundamental curves, for otherwise F is rational (Lemma 10.1) and there is nothing to prove.

Under these assumptions a very short proof of the rationality of F can be given, *provided the characteristic p is different from* 2. The proof is as follows:

The irreducible net $|K_a|$ defines a rational transformation $T: F \to S_2$ of F onto a projective plane, the T-transform of $|K_a|$ being the net of lines in S_2. Since $|K_a|$ has no base points, T is regular on F. Since $|K_a|$ has degree 2, the functions field Σ $(= k(F))$ is an algebraic extension, of degree 2, of the subfield $\Omega = k(S_2)$. Since $|K_a|$ has no fundamental curves, T^{-1} has no fundamental points in S_2. Since F is non-singular, it follows that F *is the normalization of S_2 in the field* Σ (F is a "double plane"). Since $p \neq 2$, the quadratic extension $\Sigma/k(S_2)$ is separable, and if x, y are affine coördinates in S_2, then the portion of F which lies over the affine (x,y)-plane is biregularly equivalent to a surface given by an equation of the form $z^2 = f(x,y)$, where $f(x,y)$ is a polynomial *free from multiple factors.* The curve $f(x,y) = 0$ is the affine part of the branch curve Δ of the double plane F. It follows from the equation $z^2 = f(x,y)$ that if l is a line in S_2 and C is the corresponding cycle in $|K_a|$ then a singular point of C neces-

sarily corresponds to a multiple intersection of l with Δ. It follows that the general cycle in $|K_a|$ is a *non-singular* curve and is therefore of effective genus 1 (since $p(K_a)=1$). Since in an elliptic function field any function of order 2 has 4 branch points (provided the characteristic p is $\neq 2$), it follows that the branch curve Δ is of order 4. There exists at least one line l_0 in S_2 *which has no simple intersections with* Δ (this is an elementary fact from the theory of plane quartic curves). Then the corresponding cycle C_0 of $|K_a|$ will either have two double points or a tacnode (according as l_0 has two distinct intersections with C_0, both of multiplicity 2, or a single intersection of multiplicity 4); in both cases C_0 has the "equivalent" of *two* double points and therefore degenerates, i.e., C_0 is not a prime cycle. A prime component of C_0 cannot be linearly equivalent to an integral multiple of C_0, and therefore F is rational, by Proposition 9.1.

The above proof is not applicable to the case $p=2$. We shall now give another proof which is valid for any characteristic. This second proof is "constructive," for in it the surface F is explicitly constructed as a member of an algebraic system of surfaces having $p_a=P_2=0$ and $(K^2)=2$.

By Lemma 10.2 we have

$$\dim|2K_a|=6, \qquad \dim|4K_a|=20. \tag{2}$$

We fix a prime cycle C_0 in $|K_a|$, defined over k. The functions ξ in Σ such that $(\xi)+C_0>0$ form a vector space (over k) of dimension 3 (since $\dim|K_a|=2$), and 1 is one such function. We fix two elements x, y in Σ such that $\{1, x, y\}$ is a basis of that vector space. From what was said above in the course of the proof of rationality of F in the case $p\neq 2$, Σ is a quadratic extension of $k(x, y)$, and x, y are affine coördinates of the projective plane S_2 of which F is the normalization (in Σ). Clearly, Σ is a simple extension of $k(x, y)$. We shall now introduce a particular primitive element of $\Sigma/k(x, y)$.

The functions ξ in Σ such that $(\xi)+2C_0>0$ form a vector space (over k) of dimension 7 (since $\dim|2K_a|=6$, by (2)). The 6 functions $1, x, y, x^2, xy, y^2$ belong to that space and are linearly independent over k. Let z be a function in that vector space which is linearly independent of the above 6 functions. *We assert that z is a primitive element of $\Sigma/k(x, y)$.*

We first show that *z is an integral function of x and y.* For this purpose we have only to show that if $\mathfrak{p}$ is any algebraic place of Σ/k such that $\mathfrak{p}(x)$ and $\mathfrak{p}(y)$ are both finite then also $\mathfrak{p}(z)$ is finite. Let P be the center of $\mathfrak{p}$ on F. It will be sufficient to show that $P\notin C_0$, since C_0 is the only polar prime divisor of the function z. Assume the contrary: $P\in C_0$. Let

$(x) = C_1 - C_0$, $(y) = C_2 - C_0$, where $C_1, C_2 \in |K_a|$. Since $\mathfrak{p}(x) \neq \infty$, $\mathfrak{p}(y) \neq \infty$ and since $P \in C_0$, P necessarily belongs to C_1 and C_2. Since C_0, C_1, C_2 are three linearly independent cycles of the net $|K_a|$, P necessarily belongs to every cycle in $|K_a|$, i.e., P is a base point of $|K_a|$, in contradiction with the fact that $|K_a|$ has no base points.

To show that z is a primitive element of $\Sigma/k(x,y)$ we have only to show that $z \notin k(x,y)$. Assume the contrary. Then z is a polynomial $f(x,y)$ in x, y, since z is an integral function of x, y. Let n be the degree of the polynomial f. *We shall show that* $n \leqq 2$, and this will produce the desired contradiction, since the 7 functions $1, x, y, x^2, xy, y^2, z$ are linearly independent over k (by our choice of z).

Assume $n > 2$ and let $f_n(x,y)$ be the sum of terms of $f(x,y)$ which are of degree n. Since $f(x,y) - f_n(x,y)$ has order $\geqq -(n-1)$ on C_0 and $f(x,y)$ has order -2 on C_0, the quotient $f_n(x,y)/x^n$ must have positive order at the prime divisor C_0 (since $n > 2$). Hence, if we denote by c the C_0-residue of y/x then $f_n(1,c) = 0$, i.e., c is algebraic over k, and so $c \in k$ (note that $c \neq 0, \infty$ since both x and y have order -1 on C_0). The function $y - cx$ does not, therefore, become infinite at the prime divisor C_0 and thus has no polar prime divisors at all. Hence $y - cx \in k$, in contradiction with the linear independence of $1, x, y$ over k.

We have thus established that z is a primitive element of $\Sigma/k(x,y)$ (and, moreover, that it is an integral function of x, y).

To find the irreducible equation for z over $k(x,y)$ we consider the 22 monomials $x^m y^n$ $(0 \leqq m+n \leqq 4)$, $x^m y^n z$ $(0 \leqq m+n \leqq 2)$ and z^2, where m and n are non-negative integers. If ω is any of these monomials then $(\omega) + 4C_0 > 0$. Since $\dim |4K_a| = 20$ and since therefore the vector space of the functions ξ such that $(\xi) + 4C_0 > 0$ has dimension 21, it follows that *the above 22 monomials are linearly dependent over* k. The first 21 of those monomials (i.e., those which are different from z^2) are linearly independent over k (since x and y are algebraically independent over k and since $z \notin k(x,y)$). Hence the relation of linear dependence between the above 22 monomials has the following form:

$$z^2 + g_2(x,y)z + g_4(x,y) = 0, \tag{3}$$

where g_2 and g_4 are polynomials in x, y, of degree not greater than 2 and 4 respectively. The equation (3) is irreducible (since $z \notin k(x,y)$). We shall denote by g the polynomial on the left-hand side of (3) as well as the (projective) surface in S_3 defined by the equation (3). The surface g is birationally equivalent to F since $\Sigma = k(x,y,z)$. *We have therefore to prove the rationality of the surface* g.

We shall now make three remarks.

Remark 1. We denote by g_a the affine part of the surface, i.e., the set of points of g which are at finite distance in the (x,y,z)-space. We assert that *the affine surface g_a is normal.* To see this we have to show that $k[x,y,z]$ is the integral closure of $k[x,y]$ in Σ $(=k(x,y,z))$; or equivalently, that $\{1,z\}$ form an integral basis, over $k[x,y]$, of the integral closure of $k[x,y]$ in $k(x,y,z)$. Let ω be any function in $k(x,y,z)$ which is integral over $k[x,y]$. Since C_0 is the only polar curve of x and y on F, C_0 is also the only polar curve of ω on F. Hence there exists an integer $n \geqq 1$ such that $(\omega) + nC_0 \geqq 0$ (on F). Now, we have $\dim |nK_a| = n(n+1)$ [see (1)], and hence the vector space V of all functions ω in Σ such that $(\omega) + nC_0 \geqq 0$ (on F) has dimension $n(n+1)+1$. On the other hand, the functions ζ of the form $f_n(x,y) + zf(x,y)$, where f_a and f_{n-2} are polynomials in x,y, of degree not greater than n and $n-2$ respectively, belong to that vector space. Since x,y are algebraically independent and since $z \notin k(x,y)$, the dimension of the space of functions ζ is $(n+1)(n+2)/2 + (n-1)n/2 = n(n+1)+1$. Hence this vector space coincides with V. Thus ω has the above form $f_n(x,y) + zf_{n-2}(x,y)$, and this proves our assertion.

Thus g_a is the normalization of the affine (x,y)-plane in the field Σ. Since F is the normalization of the *projective* (x,y)-plane in Σ, it follows that—up to biregular equivalence—the affine surface g_a can be identified with the part of the surface F which consists of those points P at which x and y are both finite (i.e., the point P of F such that $x,y \in o_P$). Since $(x) = C_1 - C_0$, $(y) = C_2 - C_0$, it follows that the part of F in question is $F - C_0$. *Thus g_a can be identified with $F - C_0$.*

If we now set $x' = 1/x$, $y' = y/x$, $z' = z/x^2$, then we find $(x') = C_0 - C_1$, $(y') = C_2 - C_1$, $(z') + 2C_1 > 0$. The three functions x', y', z' are connected by the equation

$$z'^2 + g'_2(x',y')z' + g'_4(x',y') = 0, \tag{3'}$$

where $g'_2(x',y') = x'^2 g_2(1/x', y'/x')$, $g'_4(x',y') = x'^4 g_4(1/x', y'/x')$. If we denote by g' the (projective) surface defined by (3') and by g'_a the affine part of that surface (in the affine (x,y,z)-space), then we have, by a similar argument, that *the affine surface g'_a can be identified with $F - C_1$.*

Finally, if we set $x'' = x/y$, $y'' = 1/y$, $z'' = z/y^2$, then x'', y'', z'' are connected by the equation

$$z''^2 + g''_2(x'',y'')z'' + g''_4(x'',y'') = 0, \tag{3''}$$

where $g''_2(x'',y'') = y''^2 g_2(x''/y'', 1/y'')$, $g''_4(x'',y'') = y''^4 g_4(x''/y'', 1/y'')$.

and the *affine* surface g''_a defined by (3″) can be identified with $F - C_2$. Since C_0, C_1, C_2 have no common points, it follows that *the three affine surfaces* g_a, g'_a, g''_a *cover* F.

Remark 2. The surface g is of order 4, unless g_4 is of degree $\leqq 3$ in which case g is of order $\leqq 3$. In the latter case the surface g is certainly rational since it cannot be an elliptic cubic cone (the birationally equivalent surface F having arithmetic genus zero). If g is a quartic surface, the point O at infinity on the z-axis is a double point of g. The plane sections of g passing through O have a tacnode at O (i.e., they have a double point infinitely near the double point O). This is easily seen by a direct analysis of the singular point O. [In the case of characteristic zero the general plane section of g through O is an elliptic quartic curve, since it corresponds birationally to a general anticanonical curve K_a on F. Since g has no singularities at finite distance (in view of the inclusion $g_a \subset F$) the general plane section of F has no singular points outside of O, and this shows again that O must be a tacnode of the section.] In other words, O is a tacnodal point of the surface g. The plane at infinity is the tangent plane of g at O, and the section of g with that plane decomposes into 4 lines through O (these are obtained by equating to zero the highest degree homogeneous component g_{44} of the polynomial $g_4(x, y)$). Conversely, it is easily seen that if a quartic surface in S_3 has a tacnodal point O then the equation of that surface takes the form (3) provided we take O as the point at infinity on the z-axis and take the tangent plane of the surface at O as the plane at infinity. The proof developed in this section will therefore establish the fact that *any quartic surface in* S_3 *which has a tacnodal point is either rational or is birationally equivalent to the ruled surface of genus* 1 (*say, an elliptic cubic cone*).

Remark 3. The linear system L on F which corresponds to the system of plane sections of g is defined by the 4 functions $1, x, y, z$. Since we have on F: $(x) = C_1 - C_0$, $(y) = C_2 - C_0$ and $(z) = D - 2C_2$, where $D \in |2K_a|$, L is generated by the 4 (linearly independent) cycles $2C_0, C_1 + C_0, C_2 + C_0, D$. (Note that L has no fixed components, since the *irreducible* curve C_0 cannot be a component of D; otherwise D would be of the form $C_0 + C_3$, where $C_3 \in |K_a|$, and D would belong to the system generated by $2C_0, C_1 + C_0$, $C_2 + C_0$, and this is not the case.) The base points of L are the 4 (generally distinct) intersections of C_0 and D. These intersections are the fundamental points of the birational transformation $T: F \to g$. To each of these points corresponds on g one of 4 lines in which the plane at infinity cuts g. The

irreducible curve C_0 is a fundamental curve of L (since C_0 is a fixed component of the two-dimensional subsystem of L which is generated by the cycles $2C_0, C_1 + C_0, C_2 + C_0$). This curve C_0 is the only exceptional curve of T and to it corresponds on g the point O, which is thus a fundamental point of T^{-1}.. It is easily checked that O is the only fundamental point of T^{-1} and that the intersections of C_0 with D are the only fundamental points of T.

Note that the above considerations show that with our choice of C_0 (C_0-*irreducible*) the surface g must be of order exactly 4 (and not less), for L—a subsystem of the system $|2K_a|$, which has degree 8—has exactly 4 base points (or the equivalent of 4 base points if C_0 and D have some multiple intersections). The same conclusion holds true if C_0 is reducible, provided C_0 and D have no common components. Suppose, however, that C_0 is reducible and that one of its irreducible components, say Δ, is also a component of D. Then Δ is a fixed component of L, and the defining linear system of the birational transformation $T: F \rightarrow g$ is not L but the system L_1 obtained from L by deleting the fixed component Δ from all the cycles in L (we assume here, for simplicity, that L_1 has no fixed component). In this case it is easily found that L_1 has degree < 4, whence g is a surface of order $\leqq 3$. In this case, then, the rationality of F follows at once. We observe, however, that since we have assumed F to be a relatively minimal model, the rationality of F follows already, in view of Lemma 9.1, from the existence of a cycle $C_0 \in |K_a|$ which is not prime. (End of the remarks.)

As was stated at the end of Remark 2, we propose to prove in this section the following proposition:

PROPOSITION 10.3. *Every irreducible quartic surface g given by an equation of the form* (2) *is birationally equivalent to a cubic surface in S_3 (or to a surface of order < 3).*

This will establish Castelnuovo's criterion of rationality in the case $(K^2) = 2$, since every cubic surface, other than an elliptic cubic cone, is rational and since F—which has arithmetic genus zero—cannot be birationally equivalent to an elliptic cubic cone.

The intersection of the surface g with a plane is a cycle of degree 4. The proof of Proposition 10.3 will be based on the following lemma:

LEMMA 10.4. *If there exists a plane π through the tacnodal point O of g different from the plane at infinity, such that the intersection $g \cdot \pi$ splits into two cycles of degree 2, both passing through O, then the surface g is birationally equivalent to a surface of order $\leqq 3$.*

Proof. Let p_1, p_2, p_3, p_4 be the 4 lines in which the surface g is cut by the plane at infinity (the lines p_i need not be distinct). Subject to an affine change of the nonhomogeneous coördinates x and y, we may assume that π is the plane $x=0$. Setting $g_2(0,y)=h_2(y)$, $g_4(0,y)=h_4(y)$, we see that the intersection cycle $g\cdot\pi$ consists of the cycle Γ defined by the equation

$$z^2+h_2(y)z+h_4(y)=0 \tag{4}$$

and perhaps of one (and only one) of the line p_i (counted to a suitable multiplicity). If $h_4(y)$ is of degree >2 then equation (4) must be reducible, for otherwise Γ would be a prime cycle of degree $\geqq 3$, and the intersection cycle $g\cdot\pi$ could not split into two cycles of degree 2. Assume now that $h_4(y)$ is of degree $\leqq 2$. Then Γ is a cycle of degree 2, and Γ *does not pass through the point* O (the point at infinity on the z-axis). The intersection cycle $g\cdot\pi$ is then of the form $\Gamma+2p_1$ (where p_1 is one of the lines p_i). Since $O\notin\Gamma$, $g\cdot\pi$ could not be a sum of two cycles of degree 2, both passing through O, unless Γ splits into two lines q_1, q_2 (in which case $g\cdot\pi$ has the desired decomposition $(q_1+p_1)+(q_2+p_1)$). Hence also in this case equation (4) must be reducible. Let then

$$z^2+h_2(y)z+h_4(y)=(z+\psi(y))(z+\psi'(y)),$$

where $\psi(y)$ and $\psi'(y)$ are polynomials of degree $\leqq 2$. We now replace z by the element $z+\psi(y)$ (this amounts to a birational transformation $x'=x$, $y'=y$, $z'=z+\psi(y)$).[7] Then the equation (3) takes the form

$$z^2+\bar{g}_2(x,y)z+x\bar{g}_3(x,y)=0,$$

where $\bar{g}_2(x,y)$ and $\bar{g}_3(x,y)$ are polynomials of degree at most 2 and 3 respectively. Thus we may assume that *in the original equation* (3) *the polynomial* $g_4(x,y)$ *is divisible by* x. But then the birationally equivalent surface g' defined by equation (3′) (Remark 1) is a surface of degree $\leqq 3$ (since $g'_4(x',y')$ has now degree $\leqq 3$). This establishes Lemma 10.4.

The surface g belongs to an irreducible algebraic system N/k of quartic surfaces in S_3 whose general member G is defined by an equation of the form similar to (3):

$$G\colon z^2+G_2(x,y)z+G_4(x,y)=0,$$

[7] We observe—going back to the original surface F—that the conditions imposed on the element z (i.e., the divisor of the function z to be a mutiple of $-2C_0$ and z to be linearly independent of the functions $1, x, y, x^2, xy, y^2$) do not determine z uniquely. Any function of the form $cz+\psi(x,y)$, where $0\neq c\in k$ and $\psi(x,y)$ is any polynomial of degree $\leqq 2$, will satisfy the same conditions.

where $G_2(x,y)$ and $G_4(x,y)$ are polynomials of degree 2 and 4 respectively, with coefficients which are algebraically independent over k. Note that $\dim N = 21$.

We shall prove that *there exists a plane σ such that σ and the intersection cycle $G\cdot\sigma$ satisfy the conditions of Lemma* 10.4 (*with g and π replaced by G and σ*). We assume for a moment that this assertion has already been proved and we show how Projosition 10.3 can be derived from this assertion.

The surface g is a specialization of G, over k. Let $(G,\sigma)\xrightarrow{k}(g,\pi)$ be an extension of the specialization $G\xrightarrow{k}g$. Then π is a plane through O, and since $G\cdot\sigma$ is a sum of two cycles of degree 2 passing through O, the same is true of the intersection cycle $g\cdot\pi$ (note that O is rational point over k). If π is not the plane at infinity, then Proposition 10.3 is established, in view of Lemma 10.4. If, however, π is the plane at infinity, then we replace both G and g by their birational transforms G' and g' under the transformation $x'=1/x,\ y'=y/x,\ z'=z/x^2$. Then also G' is a general member of N/k. Furthermore, if the equation of σ is $ax+by+c=0$ we denote by σ' the plane $a+by'+cx'=0$ (the transform of σ under the transformation $x'=1/x,\ y'=y/x$) and by π' the plane $x'=0$. It is clear that σ' and G' are in the same relationship to each other as are σ and G (i.e., the conditions of Lemma 10.4 are satisfied when g and π are replaced by G' and σ') and that (g',π') is a specialization of (G',σ') over k. Since π' is not the plane at infinity, g'—and therefore also g—is birationally equivalent to a surface of order $\leqq 3$. This completes the proof of Proposition 10.3.

We now proceed to the proof of the assertion that there exists a plane σ such that σ and the cycle $G\cdot\sigma$ satisfy the conditions of Lemma 10.4 (with g and π replaced by G and σ respectively).

The algebraic system N, of which G is a general member over k, contains an irreducible subsystem N' whose general member over k is the surface H defined by

$$(5)\qquad H:[z+\phi_2(x,y)]^2+A_2(x,y)[z+\phi_2(x,y)]+A_1(x,y)A_3(x,y)=0,$$

where A_1, A_2, A_3 and ϕ_2 are polynomials in x,y, of degree $1,2,3$ and 2 respectively, with coefficients which are algebraically independent quantities over k. We shall denote also by H the left-hand side of the equation (5). Then

$$(6)\qquad H=z^2+H_2(x,y)z+H_4(x,y),$$

where

$$H_2(x,y) = A_2(x,y) + 2\phi_2(x,y), \tag{7}$$

$$H_4(x,y) = A_1(x,y)A_3(x,y) + A_2(x,y)\phi_2(x,y) + [\phi_2(x,y)]^2, \tag{8}$$

which shows that H is indeed a specialization of G over k and that consequently $N' \subset N$. Now, the surface H satisfies the conditions of Lemma 10.4 (with g replaced by H and π replaced by the plane $A_1(x,y) = 0$), for the intersection of H with the plane $A_1(x,y) = 0$ splits into two cycles of degree 2, defined respectively by the equations $z + \phi_2(x,y) = 0$ and $z + \phi_2(x,y) + A_2(x,y) = 0$, and both these cycles pass through the tacnodal point O. So everything will be proved if we show that *also the surface H is (as well as G) a general member of N/k* (and that therefore N' actually coincides with N).

From now on we shall denote by G^* (resp., H^*) the point, in the affine space of dimension 21, whose (non-homogeneous) coördinates are the coefficients of $G_2(x,y)$ and $G_4(x,y)$ (resp., of $H_2(x,y)$ and $H_4(x,y)$). We have $\dim G^*/k = 21$, *and what we have to prove is that also*

$$\dim H^*/k = 21. \tag{9}$$

We shall denote by A^* (resp., ϕ^*) the point in the affine space of dimension 25 (resp., of dimension 6) whose (non-homogeneous) coördinates are the coefficients of the polynomials $A_1(x,y)$, $A_2(x,y)$, $A_3(x,y)$, $\phi_2(x,y)$ (resp., of the polynomial $\phi_2(x,y)$). We have $\dim A^*/k = 25$, and we also know, by (7) and (8), that the point H^* is rational over the field $k(A^*)$. *Hence, in order to prove* (9), *we have to show* that

$$\dim A^*/k(H^*) = 4. \tag{10}$$

Since we have at any rate $\dim H^*/k \leqq 21$, it follows that $\dim A^*/k(H^*) \geqq 4$. So *all we have to prove is that*

$$\dim A^*/k(H^*) \leqq 4. \tag{11}$$

Let A_1^* denote the point, in the affine 3-space, whose (non-homogeneous) coördinates are the coefficients of the linear polynomial $A_1(x,y)$ (note that this point is also rational over $k(A^*)$). We show first that

$$\dim A^*/k(H^*, A_1^*) \leqq 3. \tag{12}$$

We fix a (finite) specialization $\bar{A}^*$ of A^* over $k(H^*, A_1^*)$ *which is algebraic over* $k(H^*, A_1^*)$. The coördinates of the point $\bar{A}^*$ are the coefficients of certain polynomials $\bar{A}_1(x,y)$, $\bar{A}_2(x,y)$, $\bar{A}_3(x,y)$ and $\bar{\phi}_2(x,y)$ (of indicated

degrees). We have $\bar{A}_1(x,y) = A_1(x,y)$ since $\bar{A}^*$ is a specialization of A^* over $k(A_1^*)$. By (7) and (8) we have

$$H_2(x,y) = \bar{A}_2(x,y) + 2\bar{\phi}_2(x,y),$$

$$(13) \qquad H_4(x,y) = A_1(x,y)\bar{A}_3(x,y) + \bar{A}_2(x,y)\bar{\phi}_2(x,y) + [\bar{\phi}_2(x,y)]^2,$$

whence upon setting

$$(14) \qquad q_2(x,y) = \bar{\phi}_2(x,y) - \phi_2(x,y),$$

we find, again by (7) and (8):

$$(15) \qquad A_2(x,y) = \bar{A}_2(x,y) + 2q_2(x,y),$$

$$(16) \qquad A_1(x,y)A_3(x,y) = A_1(x,y)\bar{A}_3(x,y) + \bar{A}_2(x,y)q_2(x,y) + [q_2(x,y)]^2.$$

From (16) it follows that $q_2(x,y)$ must satisfy the congruence

$$\bar{A}_2(x,y)q_2(x,y) + [q_2(x,y)]^2 \equiv 0 \pmod{A_1(x,y)}.$$

Since q_2 must be a polynomial of degree $\leqq 2$ this congruence has only two general solutions q_2^*; they are given by $q_2^*(x,y) = A_1(x,y)p_1(x,y)$ and $q_2^*(x,y) = -\bar{A}_2(x,y) + A_1(x,y)p_1(x,y)$ respectively, where $p_1(x,y)$ is the general linear polynomial in x,y. Each of these solutions depends on three arbitrary parameters; more precisely, we have $\dim q_2^*/k(H^*,A_1^*) = 3$ (note that the coefficients of $\bar{A}_2(x,y)$ are algebraic over $k(H^*,A_1^*)$, by our choice of the specialiaztion $\bar{A}^*$). From (14), (15) and (16) we see that the coefficients of ϕ_2, A_2 and A_3 are rational over $k(\bar{A}^*,q_2^*)$. Since $\bar{A}^*$ is algebraic over $k(H^*,A_1^*)$, it follows that A^* is algebraic over $k(H^*,A_1^*,q_2^*)$. The inequality (12) now follows from the equality $\dim q_2^*/k(H^*,A_1^*) = 3$.

In view of (12), the inequality (11) will be established if we show that

$$(17) \qquad \dim A_1^*/k(H^*) \leqq 1.$$

It will be sufficient to show that the number of *planes* $\bar{A}_1(x,y) = 0$ (in the ambient 3-space of the surface H) such that the $\bar{A}_1(x,y)$ is a (finite) specialization (over $k(H^*)$, of the linear polynomial $A_1(x,y)$), is finite.

Now, any such plane shares with the plane $A_1(x,y) = 0$ the property that its intersection with the surface H splits into two cycles of degree 2, both passing through the tacnodal point 0 of H. Now, let $\bar{A}_1(x,y)$ be such a plane. The quadratic polynomial H in z factors, $\bmod \bar{A}_1$, into a product $(z+\bar{\psi})(z+\bar{\psi}')$, where $\bar{\psi}$ and $\bar{\psi}'$ are polynomials in x,y, of degree 2 (see

proof of Lemma 10.4). We have seen in the proof of Lemma 10.4 that the birational transformation

$$\bar{x} = x/\bar{A}_1, \qquad \bar{y} = y/\bar{A}_1, \qquad \bar{z} = z + \psi/\bar{A}_1^2 \tag{18}$$

transforms H into the cubic surface

$$C: \bar{z}^2 + C_2(\bar{x}, \bar{y})z + C_3(\bar{x}, \bar{y}) = 0,$$

where C_2 and C_3 are polynomials of degree 2 and 3 respectively. From the fact that A_1, A_2, A_3 and ϕ_2 are polynomials whose coefficients are algebraically independent over k it follows at once that C is the *general cubic surface* in S_3 (referred to a special system of coödinates). Now, it is clear that in the birational transformation (18) the system of plane section of H through O corresponds to the system of plane sections of C passing through the point at infinity on the $\bar{z}$-axis. If a plane π passing through O, different from the plane at infinity and the plane $\bar{A}_1 = 0$, intersects H in a sum of two cycles, of degree 2, both passing through O, then the section of C with the corresponding plane $\alpha\bar{x} + \beta\bar{y} + \gamma = 0$ is reducible and therefore contains a line of the cubic surface. Since the general cubic surface contains only a finite number of lines, the number of planes π with the above indicated property is finite. This completes the proof.

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REFERENCES.

[1] Y. Akizuki and H. Matsumura, "On the dimension of algebraic systems of curves with nodes on a non-singular surface," *Mem. Coll. Sci. Univ. Kyoto*, series A, vol. 30, n. 2 (1957), pp. 143-150.

[2] F. Enriques, *Le superficie algebriche*, Bologna, Zanichelli, 1949.

[3] H. Matsumura and M. Nagata, "On the algebraic theory of sheets of an algebraic variety," *Mem. Coll. Sci. Univ. Kyoto*, series A, vol. 30, n. 2 (1957), pp. 157-164.

[4] Y. Nakai, "A property of an ample linear system on a non-singular variety," *ibid.*, series A, vol. 30, n. 2 (1957), pp. 151-156.

[5] ———, "On the characteristic linear systems of algebraic families," forthcoming in the *Illinois Journal of Mathematics*.

[6] O. F. G. Schilling and O. Zariski, "On the linearity of pencils of curves on algebraic surfaces," *American Journal of Mathematics*, vol. 60 (1938), pp. 320-324.

[7] F. Severi, *Vorlesungen über algebraische Geometrie*, Teubner, Berlin, 1921.

[8] C. C. Tsen, "Divisionalgebren über Funktionenkörpern," *Nach. Ges. Wiss.*, Göttingen,1933.

[9] A. Weil, "Variétés abéliennes et courbes algébriques," *Actual. Scient. et Indust.*, n. 1064, Paris, 1948.

[10] ———, "Sur les critères d'équivalence en géométrie algébrique," *Math. Ann.*, vol. 128 (1954), pp. 95-127.

[11] O. Zariski, "Algebraic surfaces," *Ergebnisse der Mathematik und ihrer Grenzgebiete* 3, n. 5 (1935).

[12] ———, "Reduction of the singularities of algebraic three-dimensional varieties," *Annals of Mathematics*, vol. 45 (1944), pp. 472-542.

[13] ———, "Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields," Mem. Amer. Math. Soc., n. 5 (1951).

[14] ———, "Introduction to the problem of minimal models in the theory of algebraic surfaces," forthcoming in the *Publications of the Mathematical Society of Japan* (referred to as IP).

THE DETERMINATION OF PARACOMPACTNESS BY UNIFORMITIES.*[1]

By H. H. Corson.

1. Introduction. In the theory of paracompact spaces the global nature of the concept of paracompactness is emphasized by many theorems. It would seem then that some convenient formulation would be possible in terms of uniform structures.

A possible solution to this problem was suggested by Kelley in [3, p. 208], where he conjectured that if X is topologically complete, plus another condition, then X is paracompact. This conjecture was shown by Isaac Namioka to be false. However, in the theorem proved below it will be seen that a property very similar to topological completeness is successful in this rôle. In fact X is paracompact if and only if X admits a uniformity under which every filter satisfying a Cauchy-like condition has a cluster point.

A well known open question which appears to be closely related to our theorem is this. If a topological space has the property that its product with every compact space is normal, is it paracompact? Dieudonné proved in [2] that every paracompact space X has this property, and also that the set of all neighborhoods of the diagonal is a uniformity for X. Our theorem demonstrates that, together at least, these two properties are sufficient as well as necessary.

However, since paracompact is a purely topological term, we give a formulation of our results in topological terminology. As might be expected from the theory of uniform structures, this last equivalence is between paracompact spaces and spaces on which enough continuous functions are defined, these functions taking values in metric spaces (metrizable spaces).

The last section of the paper is devoted to relating our theorem with known results. Some shortcomings of our methods are pointed out, as well as some applications. An example of the latter is the use of our theorem to

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compare paracompact spaces with topological spaces which are Lindelöf spaces, Q, or topologically complete.

2. The theorem. Before stating the theorem, we should first mention that the necessary termininology and theory of filters, uniform structures and paracompact spaces may be found either in Bourbaki [1] or Kelley [3]. We will also need to define the following property of a filter, which is slightly weaker than the Cauchy property.

Definition. A filter $\mathfrak{F}$ in a uniform space $(X, \mathfrak{U})$ is *weakly Cauchy* if for every $U \in \mathfrak{U}$ some filter stronger that $\mathfrak{F}$ becomes U small. That is there is a filter $\mathfrak{H}_U$, $\mathfrak{F} \subset \mathfrak{H}_U$, and $H \times H \subset U$ for some $H \in \mathfrak{H}_U$. All topological spaces will be assumed to be Hausdorff spaces.

THEOREM 1. *For a topological space X the following are equivalent:*

(a) *X is paracompact.*

(b) *The set of all neighborhoods of the diagonal is a uniformity for X, and the product of X with every compact space is normal.*

(c) *There is a uniformity for X under which every weakly Cauchy filter has a cluster point.*

(d) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ has a cluster point in X.*

Proof of (a) $\Rightarrow$ (*b*). This follows from the theorem of Dieudonné [2].

Proof of (b) $\Rightarrow$ (c). Assuming (b) we may choose a uniformity $\mathfrak{U}$ for X where $\mathfrak{U}$ is the set of all neighborhoods of the diagonal Δ. Suppose $\mathfrak{F}$ is a filter which is weakly Cauchy under $\mathfrak{U}$ and such that $\mathfrak{F}$ has no cluster point in X. Let A be the cluster points of $\mathfrak{F}$ in some compactification of X, say $\alpha(X)$. Then it is easily verified that Δ and $A \times X$ are disjoint closed sets in $\alpha(X) \times X$, and consequently by (b) there is a neighborhood U of Δ such that closure (U) does not intersect $A \times X$.

By our choice of $\mathfrak{U}$ we have that $U \in \mathfrak{U}$. Since $\mathfrak{F}$ is assumed weakly Cauchy, there is a filter $\mathfrak{H}$ stronger than $\mathfrak{F}$ with $H \in \mathfrak{H}$ and $H \times H$ contained in U. We now have this situation. $\mathfrak{H}$ has a cluster point in $\alpha(X)$, this cluster point is also a cluster point of $\mathfrak{F}$, and it clearly is not in A. This contradicts the assumption that A was the set of all cluster points of $\mathfrak{F}$.

Proof of (c) $\Rightarrow$ (*d*). Let $\mathfrak{F}$ be a filter whose image has a cluster point in every metric continuous image of X. We assume that $\mathfrak{U}$ has the property stated in (c) and prove that $\mathfrak{F}$ is weakly Cauchy with respect to $\mathfrak{U}$.

If $U \in \mathfrak{U}$, then we must show there is an $\mathfrak{H}$ stronger than $\mathfrak{F}$ such that $\mathfrak{H}$ becomes U small. However, by the characterization of $\mathfrak{U}$ in terms of écarts, it follows that there is a pseudo metric p with $U' = \{(x, y) : p(x, y) \leqq 1\}$ contained in U. Furthermore, if $\mathfrak{T}$ is the original topology for X, the identity map from $(X, \mathfrak{T})$ to (X, p) is continuous; thus if Y is the metric space canonically associated with (X, p) then there is an induced map f from $(X, \mathfrak{T})$ to Y.

From our assumptions on $\mathfrak{F}$, $f(\mathfrak{F})$ clusters in Y; therefore there is a filter $\mathfrak{H}'$ stronger than $f(\mathfrak{F})$ which converges in Y. It follows that $\mathfrak{H} = \sup\{f^{-1}(\mathfrak{H}'), \mathfrak{F}\}$ is a filter which has the properties required. In fact since $\mathfrak{H}'$ is Cauchy in Y, there is a $H' \in \mathfrak{H}'$ such that $H' \times H'$ is contained in $f \times f(U')$. Thus for some $H \in \mathfrak{H}$, $H \subset f^{-1}(H')$ and $H \times H \subset U' \subset U$.

Proof of (d) $\Rightarrow$ (a). Suppose X were not paracompact under the conditions in (d), then there is an open covering $\mathfrak{G}$ of X with no locally finite open refinement.

Suppose $\boldsymbol{R}$ is the collection of finite subsets of $\mathfrak{G}$, and for each $\mathcal{R} \in \boldsymbol{R}$ let $F(\mathcal{R})$ be the complement of $\bigcup\{R : R \in \mathcal{R}\}$. Obviously no finite subcollection of $\mathfrak{G}$ covers X, therefore $\{F(\mathcal{R}) : \mathcal{R} \in \boldsymbol{R}\}$ is the base of a filter $\mathfrak{F}$; and, since $\mathfrak{G}$ is an open cover, $\mathfrak{F}$ has no cluster point in X.

By hypothesis there is a continuous mapping f of X into a metric space M such that $f(\mathfrak{F})$ has no cluster point in M. Let $\mathfrak{U}$ be the open cover of M defined by $U \in \mathfrak{U}$ if and only if U is the interior of the complement of some $f(F)$ with $F \in \mathfrak{F}$. Since by A. H. Stone's theorem [6] each metric space is paracompact, there is a locally finite open refinement $\mathfrak{V}$ of $\mathfrak{U}$.

We now consider the cover $\mathfrak{L}$ of X where $\mathfrak{L} = \{f^{-1}(V) : V \in \mathfrak{V}\}$. $\mathfrak{L}$ is locally finite in X since $\mathfrak{V}$ is locally finite in M. Moreover, $L \in \mathfrak{L}$ implies L is in the complement of some $\mathfrak{F}(\mathcal{R})$, so $\mathcal{R}(L) = \{L \cap R : R \in \mathcal{R}\}$ is a cover of L.

Let us show that $\mathfrak{B} = \bigcup\{\mathcal{R}(L) : L \in \mathfrak{L}\}$ is a locally finite open refinement of $\mathfrak{G}$. $\mathfrak{B}$ is obviously a cover of X, and each $L \cap R \in \mathcal{R}(L)$ is contained in some member of $\mathfrak{G}$. To show that $\mathfrak{B}$ is locally finite, fix $x \in X$ and a neighborhood U of x such that U intersects only a finite number

of members of $\mathcal{L}$. Since each of these L is covered by a finite number of $L \cap R$, we see that U intersects only a finite number of $C \in \mathcal{C}$. This contradiction completes the proof.

3. Comments.

I. A space X may be defined to be topologically complete if it is complete under the strongest admissible uniformity $\mathcal{U}$. Closely related to this (see [4] and [5]), X is Q if it is complete under the weakest uniformity $\mathcal{V}$ such that all real valued continuous functions are uniformly continuous. Since (c) of Theorem 1 may be stated in terms of $\mathcal{U}$, the following question arises. Which are the spaces such that every filter weakly Cauchy with respect to $\mathcal{V}$ has a cluster point?

THEOREM 2. *For a regular topological space X the following are equivalent:*

(a) *X is a Lindelöf space.*

(b) *Every filter weakly Cauchy with respect to $\mathcal{V}$ has a cluster point.*

(c) *If a filter $\mathcal{F}$ is such that its image has a cluster point in the reals under every real continuous image of X, then $\mathcal{F}$ has a cluster point in X.*

Proof of (a) $\Rightarrow$ (b). Let $\mathcal{F}$ be a filter weakly Cauchy under $\mathcal{V}$. Since X is a regular Lindelöf and hence normal, there is a filter $\mathcal{H}$ weaker than $\mathcal{F}$ such that (i) $\mathcal{H}$ has a base of sets of the form $Z(f) = \{x : f(x) = 0\}$ and (ii) $\mathcal{H}$ has the same cluster points as $\mathcal{F}$.

Suppose $\mathcal{F}$, and therefore $\mathcal{H}$, has no cluster point. Then the hypothesis that X is Lindelöf implies the existence of a countagle collection of $Z(f_i) \in \mathcal{H}$ such that $\cap Z(f_i)$ is empty. Since we may also assume $0 \leqq f_i \leqq 1$, we have that $g = (\sum_1^\infty 2^{-i} f_i)^{-1}$ is a real continuous function defined on all of X. Moreover, if $V = \{(x, y) : |g(x) - g(y)| \leqq 1\}$, then no filter stronger than $\mathcal{F}$ can become V small.

Proof of (b) $\Rightarrow$ (c). The proof is entirely similar to that of (c) $\Rightarrow$ (d) in Theorem 1.

Proof of (c) $\Rightarrow$ (a). Let $\mathcal{F}$ be a filter with no cluster point. By (c) there is a real map f such that $f(\mathcal{F})$ has no cluster point in the real numbers. Let $F_n \in \mathcal{F}$ be chosen with the property $|f(F_n)| \geqq n$. Then $\cap F_n$ is void, and thus X is Lindelöf.

II. Theorem 1 may be considered an extension of this known theorem (see [3], Chap. 6, Prob. T, or [1], Chap. 2, Sect. 4, Prob. 10). A locally compact space X is paracompact if (and only if) there is an entourage U in the strongest uniformity for X such that $U[x]$ is compact for every x. In fact, if a filter $\mathfrak{H}$, stronger than $\mathfrak{F}$, is eventually contained in the compact subspace $U[x]$ for some x, then $\mathfrak{H}$ and hence $\mathfrak{F}$ has a cluster point.

However, the following criterion of Kelley's should be mentioned in this connection: X is paracompact if and only if (K) given an open cover $\mathfrak{O}$ of X there is a neighborhood D of Δ such that $\{D[x] : x \in X\}$ is a refinement of $\mathfrak{O}$. One easily finds that (K) implies property (c) of Theorem 1, but the converse is perhaps most easily proved by first proving our Theorem.

III. It is obvious that every Cauchy filter is contained in a Cauchy ultrafilter, and from our theorem every filter weakly Cauchy under the strongest uniformity $\mathfrak{U}$ on a paracompact space is contained in a weakly Cauchy ultrafilter. However, it is not even true in general that every weakly Cauchy filter is weaker than a maximal weakly Cauchy filter. The next theorem illustrates that it is precisely this failure which makes our result a theorem instead of a panacea.

First let us notice that a maximal weakly Cauchy filter (under $\mathfrak{U}$) is either eventually in a set or eventually out of it, in fact it is an ultrafilter. Let A be a subset of X, $\mathfrak{F}$ a maximal weakly Cauchy filter in X, and suppose no $F \in \mathfrak{F}$ is contained in A or in its complement. Since $\mathfrak{F}$ is maximal there is a $U_1 \in \mathfrak{U}$ such that no filter stronger than $\mathfrak{F}|A$ become U_1 small; and there is a U_2 having the same property with respect to $\mathfrak{F}$ restricted to the complement of A. It follows that no filter stronger than $\mathfrak{F}$ becomes $U_1 \cap U_2$ small, and thus a contradiction.

THEOREM 3. *A topologically complete space X is paracompact if and only if every filter in X, weakly Cauchy with respect to $\mathfrak{U}$, is weaker than a maximal weakly Cauchy filter.*

The necessity is obvious, while the sufficiency follows from Theorem 1 and the above remarks. In fact an ultrafilter is Cauchy if it is weakly Cauchy.

We know from a theorem of Stone's [6] that the product P of more than a countable number of non-compact, paracompact spaces is not paracompact, but P is topologically complete. Thus there is not necessarily a supremum in a maximal tower of weakly Cauchy filters.

TULANE UNIVERSITY.

REFERENCES.

[1] N. Bourbaki, *Topologie Générale*, Paris, 1948.

[2] J. Dieudonné, "Une généralization des espaces compacts," *Journal de Mathématiques*, ser. 9, vol. 23 (1944), pp. 65-76.

[3] J. L. Kelley, *General Topology*, New York, 1955.

[4] L. Nachbin, "Topological vector spaces of continuous functions," *Proceedings of the National Academy of Sciences*, vol. 40 (1954), pp. 471-474.

[5] T. Shirota, "A class of topological spaces," *Osaka Mathematical Journal*, vol. 4 (1953), pp. 23-40.

[6] A. H. Stone, "Paracompactness and product spaces," *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 977-982.

ON THE STRUCTURE OF ORTHOGONAL GROUPS.*

By TSUNEO TAMAGAWA.

Let V be a vector space over a field k and Q a quadratic form defined on V whose associated bilinear form $B(x,y)=Q(x+y)-Q(x)-Q(y)$ is nondegenerate. Let $O(V)$ be the orthogonal group of Q and $\Omega(V)$ the commutator group of $O(V)$. If the dimension n of V is $\geqq 3$ and the index ν of Q is $\geqq 1$, the factor group of $\Omega(V)$ modulo its center is simple except in the following cases:

(i) $n=3$ and k is the field of three elements; (ii) $n=4$ and $\nu=2$.

This theorem was proved by D. E. Dickson [3] in the case of finite ground field, then by J. Dieudonné [4] in the general case. In this paper we will give another proof of this theorem based on a principle given by K. Iwasawa in his paper [6]. The author wishes to express his hearty thanks to Professor Iwasawa who read the original manuscript and gave him several important suggestions.

For our convenience, we will use the same terminology as in C. Chevalley; *The algebraic theory of spinors,* Chapter I. We will define some notations we will use in this paper. The conjugate of a subspace U will be denoted by U'. If U is nonisotropic, the restriction Q_U of Q to U is a quadratic form whose associated bilinear form is nondegenerate. We will denote the index of Q_U (sometimes to be referred to as the index of U) by $\nu(U)$, the orthogonal group of Q_U by $O(U)$ and the commutator group of $O(U)$ by $\Omega(U)$. Since every $\rho_U \in O(U)$ is extended uniquely to $\rho \in O(V)$ which induces the identity transformation on U', we can consider $O(U)$ a subgroup of $O(V)$. The subspace spanned by vectors $x_1, \cdots, x_s$ will be denoted by $\langle x_1, \cdots, x_s \rangle$. If u is a nonsingular vector, we denote the symmetry with respect to the hyperplane $\langle u \rangle'$ by $\sigma_{\langle u \rangle}$. In the case of characteristic 2 the term "symmetry" means "transvection orthogonale" defined by J. Dieudonné [4], p. 41.

1. Subgroups $H_{\langle x \rangle}$.

LEMMA 1.[1] *Let x be a singular vector and u a vector in $\langle x \rangle'$. A linear*

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[1] Orthogonal transformations of the type $\rho_{x,u}$ were introduced by M. Eichler in his

transformation $\rho_{x,u}$ of $\langle x\rangle'$ defined by $\rho_{x,u}(z)=z+B(z,u)x$ is a Q-isomorphism of $\langle x\rangle'$ onto itself which leaves x fixed, and can be extended uniquely to an orthogonal transformation of V.

Proof. It is clear that $\rho_{x,u}(x)=x$ and $\rho_{x,u}$ is a linear isomorphism of $\langle x\rangle'$ onto itself. Since $Q(u+cx)=Q(u)$ for every $u\in\langle x\rangle'$ and $c\in k$, it follows that $\rho_{x,u}$ is a Q-isomorphism. Let y be a singular vector such that $B(x,y)=1$, and U the space $\langle x,y\rangle'$. Then we have $V=\langle x,y\rangle+U$ and $\langle x\rangle'=\langle x\rangle+U$.

For every $u\in\langle x\rangle'$ there exists a $u'\in U$ such that $u-u'\in\langle x\rangle$. It is clear that $\rho_{x,u}=\rho_{x,u'}$ because $B(z,u)=B(z,u')$ for every $z\in\langle x\rangle'$, so we may assume that $u\in U$. From Witt's theorem, there exists an extension $\rho\in O(V)$ of $\rho_{x,u}$. Put $\rho(y)=ax+by+v$ with $a,b\in k$ and $v\in U$. From the relations

$$B(\rho(x),\rho(y))=B(x,y)=1;\quad B(\rho(y),\rho(z))=B(y,z)=0 \text{ for all } z\in U,$$

and $Q(\rho(y))=Q(y)=0$, we have $b=1$, $B(z,u)+B(z,v)=0$ and $a+Q(V)=0$. Then we have $\rho(y)=-Q(u)x+y-u$, hence ρ is uniquely determined.

We denote the extension ρ of $\rho_{x,u}$ by the same notation. The following properties of $\rho_{x,u}$ follow from the definition.

(i) $\rho_{x,u}\rho_{x,v}=\rho_{x,u+v}$ for $u,v\in\langle x\rangle'$.

(ii) $\rho_{x,cu}=\rho_{cx,u}$. for $0\neq c\in k$.

(iii) $\tau\rho_{x,u}\tau^{-1}=\rho_{\tau(x),\tau(u)}$ for $\tau\in O(V)$.

(iv) $\rho_{x,u}=1$ if and only if $u\in\langle x\rangle$.

It follows that the mapping $u\to\rho_{x,u}$ is an isomorphism of U into $O(V)$, and the image group depends only on the subspace $\langle x\rangle$. We will denote the subgroup by $H_{\langle x\rangle}$. From (iii), subgroups of this type are conjugate to each other. The minimum normal subgroup containing $H_{\langle x\rangle}$ will be denoted by $\Omega_1(V)$ (since $\Omega_1(V)$ is normal, the group contains all subgroups of the type $H_{\langle x\rangle}$, and is generated by those subgroups). The same notation will be used for nonisotropic subspaces U with $\dim U\geqq 3$ and $\nu(U)\geqq 1$.

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proof of our fundamental theorem (cf. M. Eichler [5] (Chap. I, § 3). However, those transformations were already observed by C. L. Siegel to define the mass of representation of 0 by an indefinite quadratic form (cf. C. L. Siegel [7], p. 408).

Let $\bar{V}$ be the $n-1$ dimensional projective space whose points are 1-dimensional subspaces of V, and C the quadric in $\bar{V}$ defined by the equation $Q(x)=0$. For every subspace U of V we denote the intersection of C with $\bar{U}$ by C_U where $\bar{U}$ is the linear subvariety in $\bar{V}$ defined by U. If G is a group of linear transformations of V, we will denote the group of projective transformations induced by G by PG. For every nonsingular linear transformation τ of V, we denote by $\bar{\tau}$ the projective transformation induced by τ. If τ is an orthogonal transformation, then $\bar{\tau}$ maps C onto itself.

We will study the behavior of the group $P\Omega_1(V)$ as a transformation group of C.

LEMMA 2. *Let $p=\langle x\rangle$ be a point on C, and T_p the hyperplane defined by $\langle x\rangle'$, namely the tangential hyperplane of C at p. Let y be a singular vector such that $B(x,y)=1$, and U the conjugate of $\langle x,y\rangle$. Then the variety $K(p)=T_p\cap C$ is the projecting cone with the vertex p and a base variety C_U, and the group $\bar{H}_p=PH_{\langle x\rangle}$ is a simply transitive transformation group of the set $C-K(p)$.*

Proof. The first part is obvious because a vector z in $\langle x\rangle'$ is singular if and only if z is a linear combination of x and a singular vector v in U.

A point $\langle z\rangle$ on C is not on $K(p)$ if and only if $B(z,x)\neq 0$, so we can normalize z so that $B(x,z)=1$. Then the mapping $z\to\langle z\rangle$ is a one-to-one mapping of the set of all singular vectors z such that $B(x,z)=1$ onto the set $C-K(p)$. Put $z=ax+by+u$ with $a,b\in k$ and $u\in U$. From $B(x,z)=1$ and $Q(z)=0$, we have $b=1$ and $a=-Q(u)$, hence we have $z=\rho_{x,-u}(y)$. The mapping $u\to\langle\rho_{x,u}(y)\rangle$ is a one-to-one (birational) correspondence of U onto $C-K(p)$, and from $\rho_{x,u}(\rho_{x,v}(y))=\rho_{x,u+v}(y)$, we can see that $\bar{H}_p$ is a simply transitive transformation group of $C-K(p)$.

Remark. If $v=1$, $K(p)$ consists of only one point p.

LEMMA 3. *Let q and r be points on C; then there exists a point p on C such that both q and r do not lie on $K(p)$.*

Proof. Let y and z be singular vectors such that $q=\langle y\rangle$ and $r=\langle z\rangle$. At first we assume that $B(y,z)\neq 0$, so we may suppose that $B(y,z)=1$. Let u be a nonsingular vector in $\langle y,z\rangle'$. Put $x=\rho_{y,u}(z)=-Q(u)y+z-u$; then we have $B(x,y)=1$, $B(x,z)=-Q(u)$, hence q and r do not lie on $K(\langle x\rangle)$. Next, we assume that $B(y,z)=0$. Then the space $\langle y,z\rangle$ is totally singular, and it is well known that there exists a singular vector x such that $B(x,y)=B(x,z)=1$. Hence we have proved our assertion.

From Lemma 2 and Lemma 3, we have the following:

LEMMA 4. *The group $P\Omega_1(V)$ is a transitive transformation group of C*

LEMMA 5. *Let p be a point on C and L_p the group of all transformations in $P\Omega_1(V)$ which leave p fixed. If $n \geqq 5$ and $\nu \geqq 2$, the group L_p acts transitively on $K(p) - p$.*

Proof. Let x, y be a pair of singular vectors such that $p = \langle x \rangle$ and $B(x, y) = 1$. Let U be the subspace $\langle x, y \rangle'$. It is clear that the group H_p and the group $P\Omega_1(U)$ are contained in L_p. Since $K(p)$ is the projecting cone with the vertex p and a base variety C_U, every point on $K(p)$ lies on a line $p \cup s$ with $s \in C_U$. From the assumption, we have $\dim U \geqq 3$ and $\nu(U) \geqq 1$; hence, from Lemma 4, the group $P\Omega_1(U)$ acts transitively on C_U. Therefore if q and r are on $K(p) - p$, there exists a $\bar{\tau} \in P\Omega_1(U)$ which maps q onto a point on the line $p \cup r$. Suppose that p, q and r are on the same line. Put $q = \langle y \rangle$ and $r = \langle z \rangle$. Then z is a linear combination $ax + by$ of x and y with $b \neq 0$. Let u be a vector in U such that $B(y, u) = a/b$. Then we have $\rho_{x,u}(y) = bz$ and $\bar{\rho}_{x,u}(q) = r$.

LEMMA 6. *Suppose that $\nu \geqq 2$. If p and q are distinct points on C, then there exists a point $r \neq p$ such that $r \in K(p)$ and $r \notin K(q)$.*

Proof. Put $p = \langle x \rangle$ and $q = \langle y \rangle$. If $\langle x, y \rangle$ is totally singular, then there exists a singular vector z such that $B(x, z) = 0$ and $B(y, z) = 1$. If $\langle x, y \rangle$ is nonisotropic, there exists a singular vector z' in $\langle x, y \rangle'$. Put $z = x + z'$. The point $r = \langle z \rangle$ satisfies our conditions in both cases.

LEMMA 7. *The group $P\Omega_1(V)$ is a primitive transformation group of C except in the case where $n = 4$ and $\nu = 2$.*

Proof. If $\nu = 1$, then from Lemma 2 and Lemma 4, the group $P\Omega_1(V)$ is 2-fold transitive, hence primitive. Suppose that $n \geqq 5$ and $\nu \geqq 2$. Let M be a subset of C with the following properties:

(i) M contains at least two points p and q.

(ii) For every $\bar{\tau} \in P\Omega_1(V)$, such that $\bar{\tau}M \cap M \neq \phi$, $\bar{\tau}M$ coincides with M.

We have only to show that M must be C itself. If the line $p \cup q$ lies on C, then from Lemma 5, both $K(p)$ and $K(q)$ are contained in M. From Lemma 6, there exists a point q' in $K(q)$ such that $q' \notin K(p)$. Hence we

may assume that $p \cup q$ does not lie on C. Then from Lemma 2, we have $M \supset C - K(p)$, $C - K(q)$. From Lemma 6, there exists a point $r \in K(p)$ such that $r \neq p$ and $r \notin K(q)$. From Lemma 5, we have $K(p) \subset M$. Hence we have $M = C$.

2. The commutator group $\mathfrak{Q}(V)$.

LEMMA 8. *The group $\Omega_1(V)$ is a subgroup of $\Omega(V)$ except in the case where $k = F_2$, $n = 4$ and $\nu = 2$.*

Proof. Since $\Omega_1(V)$ is a normal subgroup of $O(V)$, we have only to prove that $H_{\langle x \rangle}$ is a subgroup of $\Omega(V)$, where x is a singular vector. Let y be a singular vector with $B(x, y) = 1$, and U the space $\langle x, y \rangle'$. If u is a nonsingular vector in U, we have

$$\sigma_{\langle u+Q(u)x \rangle}\sigma_{\langle u \rangle}(z) = \sigma_{\langle u+Q(u)x \rangle}(z - B(u,z)u/Q(u))$$
$$= z - B(u,z)u/Q(u) - B(u + Q(u)x, z - B(u,z)u/Q(u))(u + Q(u)x)/Q(u)$$
$$= z - B(u,z)u/Q(u) + B(u,z)(u + Q(u)x)/Q(u) = \rho_{x,u}(z)$$

for every $z \in \langle x \rangle'$. From Lemma 1, we have $\rho_{x,u} = \sigma_{\langle u+Q(u) \rangle}\sigma_{\langle u \rangle}$. Since $Q(u) = Q(u + Q(u)x)$, there exists a $\tau \in O(V)$ which maps u onto $u + Q(u)x$. Therefore we have $\rho_{x,u} = \tau\sigma_{\langle u \rangle}\tau^{-1}\sigma_{\langle u \rangle} \in \Omega(V)$. Except in the case mentioned above, U is spanned by nonsingular vectors, so we have $H_{\langle x \rangle} \subset \Omega(V)$.

The following two lemmas are well known, so we omit the proof.

LEMMA 9. *Except in the case excluded in Lemma 8, the commutator group of the proper orthogonal group*[2] *$O^+(V)$ is $\Omega(V)$.* (Cf. Chevalley [2], Chapter II. 3. 9.)

LEMMA 10. *Except in the case excluded in Lemma 8, $\Omega(V)$ is generated by commutators of symmetries, and the order of every element in $O(V)/\Omega(V)$ is 2.* (Cf. Artin [1], Chapter IV, Theorem 3. 23.)

Now we will prove that $\Omega(V) = \Omega_1(V)$, except in a trivial case.

LEMMA 11. *Let x and y be a pair of singular vectors such that $B(x, y) = 1$. Then we have $O(V) = O(\langle x, y \rangle)\Omega_1(V)$.*

Proof. Let u be a nonsingular vector. Then there exists a $u' \in \langle x, y \rangle$ such that $Q(u) = Q(u')$, hence there exists an orthogonal transformation τ such that $\tau(u') = u$. Put $\tau(x) = x'$ and $\tau(y) = y'$. From Lemma 2 and Lemma 4, there exists a $\tau' \in \Omega_1(V)$ such that $\tau'(\langle x \rangle) = \langle x' \rangle$, $\tau'(\langle y \rangle) = \langle y' \rangle$.

[2] In the case of characteristic 2, a proper orthogonal transformation is defined as a product of even number of symmetries.

Since $u \in \tau(\langle x, y\rangle) = \langle x', y'\rangle$, $u'' = \tau'^{-1}(u)$ is contained in $\langle x, y\rangle$. Hence we have

$$\sigma_{\langle u\rangle} = \tau'\sigma_{\langle u''\rangle}\tau'^{-1} = \sigma_{\langle u''\rangle}(\sigma_{\langle u''\rangle}^{-1}\tau'\sigma_{\langle u''\rangle}\tau'^{-1}) \in O(\langle x, y\rangle)\Omega_1(V).$$

Since $O(V)$ is generated by symmetries except in the case mentioned above, we have proved our assertion.

LEMMA 12. *Except in the case excluded in Lemma* 8, $\Omega_1(V)$ *coincides with* $\Omega(V)$.

Proof. It is obvious that $\Omega_1(V)$ is a subgroup of $O^+(V)$, hence, from Lemma 11, we have $O^+(V) = O^+(\langle x, y\rangle)\Omega_1(V)$ and

$$O^+(V)/\Omega_1(V) = O^+(\langle x, y\rangle)/O^+(\langle x, y\rangle) \cap \Omega_1(V).$$

Since $O^+(\langle x, y\rangle)$ is an abelian group, and from Lemma 9, we have $\Omega_1(V) \supset \Omega(V)$. Therefore from Lemma 8, we have $\Omega(V) = \Omega_1(V)$.

LEMMA 13. *The commutator group of* $\Omega(V)$ *coincides with* $\Omega(V)$ *except in the following two cases:*

(i) $n = 3$, $k = F_3$; (ii) $n = 4$, $\nu = 2$, $k = F_2$ or F_3.

Proof. At first we suppose that $k \neq F_2, F_3$. Then there exists a $c \in k$ such that $c^2 \neq 0, 1$. Let x, y be a pair of singular vectors with $B(x, y) = 1$, and τ the orthogonal transformation of $\langle x, y\rangle$ which maps x onto cx and y onto $c^{-1}y$. From Lemma 10, we have $\tau^2 \in \Omega(V)$ and

$$\tau^2\rho_{x,u}\tau^{-2}\rho_{x,u}^{-1} = \rho_{c^2x,u}\rho_{x,-u} = \rho_{x,(c^2-1)u} \qquad (u \in \langle x, y\rangle').$$

Since $c^2 - 1 \neq 0$, we have $H_{\langle x\rangle} \subset \Omega(V)'$, where $\Omega(V)'$ is the commutator group of $\Omega(V)$. From Lemma 12, we have $\Omega(V)' = \Omega(V)$.

Suppose that $k = F_2$ or F_3. Then we may assume that $n \geqq 5$ or $n = 4$ and $\nu = 1$. We have only to prove that $H_{\langle x\rangle} \subset \Omega(V)'$ as in the above treated case.

At first assume that $k = F_2$. Let u be a nonsingular vector in $\langle x, y\rangle'$. From the assumption, there exists a $v \in \langle x, y\rangle'$ such that $B(u, v) = Q(v) = 1$. Then the space $\langle u, v\rangle$ is nonisotropic and of index 0. Let τ be the orthogonal transformation of $\langle u, v\rangle$ such that $\tau(u) = v$, $\tau(v) = u + v$. Then we have $\tau^3 = 1$; hence, from Lemma 10, we have $\tau \in \Omega(V)$, and

$$\tau\rho_{x,v}\tau^{-1}\rho_{x,v}^{-1} = \rho_{x,u+v}\rho_{x,v}^{-1} = \rho_{x,u} \in \Omega(V)'.$$

Since $\langle x, y\rangle'$ is spanned by nonsingular vectors, we have $H_{\langle x\rangle} \subset \Omega(V)'$.

Next, we suppose $k = F_3$. Let u be a nonsingular vector in $\langle x, y\rangle'$, then from the assumption we can find a $v \in \langle x, y\rangle'$ such that $Q(u) = Q(v)$

and $B(u,v)=0$. Then the space $\langle u,v\rangle$ is nonisotropic and of index 0. Let τ be an element of $O(\langle u,v\rangle)$ such that $\tau(u)=-v$, $\tau(v)=u$. From Lemma 10, we have $\tau^2\in\Omega(V)$ and

$$\tau^2\rho_{x,u}\tau^{-2}\rho_{x,u}^{-1}=\rho_{x,-u}\rho_{x,-u}=\rho_{x,u}.$$

Hence we have $H_{\langle x\rangle}\subset\Omega(V)'$.

3. The proof of the fundamental theorem. The following lemma is well known.

LEMMA 14. *The group $PO(V)$ is a faithful transformation group of C.* (Cf. J. Dieudonné [4], p. 30 and E. Artin [1], Theorem 3.18. The proofs given loc. cit. are valid in the case of characteristic 2 by slight modification.)

Now we can prove the simplicity of the group $P\Omega(V)$.

We exclude the following cases:

(i) $n=3$ and $k=F_3$; (ii) $n=4$ and $\nu=2$.

Let N be a normal subgroup of $P\Omega(V)$ and p a point on C. From Lemma 7, the subgroup L_p defined in Lemma 5 is maximal, and from Lemma 14, the intersection of all conjugate subgroups of L_p is $\{1\}$. It follows that if $N\neq\{1\}$, we have $NL_p=P\Omega(V)$, hence N is transitive on C. Therefore the group $N\bar{H}_p$ contains all conjugate subgroups of $\bar{H}_p$; hence, from Lemma 12, we have $N\bar{H}_p=P\Omega(V)$. Then we have $P\Omega(V)/N=\bar{H}_p/\bar{H}_p\cap N$. Since the group $\bar{H}_p$ is abelian, from Lemma 13 we have $N=P\Omega(V)$. Therefore the group $P\Omega(V)$ is simple.

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REFERENCES.

[1] E. Artin, *Geometric algebra*, New York, Interscience, 1957.

[2] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954.

[3] D. E. Dickson, *Linear groups*, Leipzig, B. C. Teubner, 1901.

[4] J. Dieudonné, *Sur les groupes classiques*, Actualités Scientifiques et Industrielles, No. 1040, 1948.

[5] M. Eichler, *Quadratische Formen und Orthogonale Gruppen*, Berlin, Springer, 1952.

[6] K. Iwasawa, "Über die Einfachheit der speziellen projectiven Gruppen," *Proceedings of the Imperial Academy of Tokyo*, vol. 17 (1941), pp. 57-59.

[7] C. L. Siegel, "Über die Zetafunktionen indefiniter Quadratischer Formen. II," *Mathematische Zeitschrift*, vol. 44 (1938), pp. 398-426.

ON THE ANALYTICITY OF THE SOLUTIONS OF ANALYTIC NON-LINEAR ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS.*[1]

By CHARLES B. MORREY, JR.

PART I. Analyticity in the Interior.

1. Introduction. In this part, we prove that any solution $u = (u^1, \cdots, u^N)$ in which each u^k is of class $C^{t_k+\mu}$, $0 < \mu < 1$, (for notations, etc., see below) of a non-linear analytic elliptic system

$$\phi_j(x, \cdots, u^k, \nabla u^k, \cdots, \nabla^{s_j+t_k} u^k, \cdots) = 0, \qquad j = 1, \cdots, N \qquad (x = (x^1, \cdots, x^\nu)) \tag{1.1}$$

of the type considered by Nirenberg and Douglis ([3], p. 532) is analytic at each interior point of its domain. The analyticity at the boundary for non-linear strongly-elliptic systems [11] will be considered in Part II.

The analyticity of any solution $u(x, y)$ of class C'''' of a single non-linear analytic elliptic equation with $\nu = 2$ was first proved in a famous memoir of S. Bernstein [1] in 1904. Other proofs of Bernstein's theorem were given by M. Gevrey [5], T. Radó [13] and H. Lewy [8] and by Bernstein himself [2]. E. Hopf [6] proved the corresponding theorem for a single equation but with any value of ν. Finally, Petrowsky [12] proved the theorem for systems which are almost as general as those considered here. Very recently, the analyticity of the solutions of *linear* elliptic systems in the interior and at the boundary has been proved by Morrey and Nirenberg [10]. Still more recently, after this paper had appeared as a report, Avner Friedman [4] extended these results and those of Part II to obtain corresponding results for quasi-analytic systems and more general systems in which the ϕ_j belong to the M_n classes of Mandebrojt [9]. His methods are entirely different from those in this paper and involve mainly bounds for the derivatives as in [10] and [5].

The proofs of Lewy, Hopf, and Petrowsky all involved extension to the

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complex domain. Lewy used his theory of hyperbolic equations to carry out the extension and Hopf's method involved the extension of certain generalized potentials to the complex domain, an idea which had been used by E. E. Levi [7]. The present paper combines this idea with a simple type of implicit function theorem in a Banach space to yield a proof which is a great deal simpler than that of Petrowsky and somewhat simpler (for the interior) than the preceding proofs except for that of Lewy and the second proof of Bernstein [2]. Of the writers mentioned above, only Morrey and Nirenberg [10], Avner Friedman [4], and Gevrey [5], considered analyticity at the boundary.

In the systems (1.1) which we consider in this paper, it is assumed that there are integers $s_1, \cdots, s_N$ and $t_1, \cdots, t_N$ such that only derivatives of order $\leqq s_j + t_k$ of u^k occur in ϕ_j. Furthermore, we assume (with Douglis and Nirenberg) that only *derivatives* of order $\geqq 0$ occur. Since we may add the same integer to all the t_j and subtract it from all the s_j, we assume (as in [3]) that

$$\max_{1 \leqq j \leqq N} s_j = 0. \tag{1.2}$$

As in [3], we say that the system (1.1) is *elliptic along a vector* $u_0(x)$ if and only if the equations of variation

$$\begin{aligned} &L_{jk}(x, D)v^k \\ &\equiv \frac{d}{d\lambda}\, \phi_j[x, \cdots, u^k + \lambda v^k, \cdots, \nabla^{s_j+t_k}(u^k + \lambda v^k), \cdots]\,|_{\lambda=0} = 0 \end{aligned} \tag{1.3}$$

form a linear elliptic system of the type in [3]. If $L^0_{jk}(x, D)$ denotes the terms of $L_{jk}(x, D)$ of order *exactly* $s_j + t_k$ ($\equiv 0$ if $s_j + t_k < 0$), the system (1.3) is said to be elliptic if and only if the determinant

$$L(x, \lambda) \equiv |\, L^0_{jk}(x, \lambda)\,| \neq 0 \text{ for real } \lambda \neq 0, \tag{1.4}$$

$L^0_{jk}(x, \lambda)$ being the characteristic polynomial of the operator $L^0_{jk}(x, D)$. It is to be noticed that this determinant is a homogeneous polynomial in λ of degree

$$m = \sum_{j=1}^{N} (s_j + t_j) \geqq 0. \tag{1.5}$$

The functions ϕ_j, u^k, and the coefficients in the L_{jk} may be complex-valued.

We use the following notations: Repeated indices are summed unless otherwise noted. If ϕ is a vector or tensor with either real or complex-valued components, $|\phi|$ denotes the square root of the sum of the squares of the components and $\|\phi\|$ denotes the square root of the sum of the squares of

the absolute values of the components; the absolute value of a single complex number z is denoted by $|z|$. The double bars will also denote norms in certain spaces to be introduced. A tensor will be of class $C^{n+\mu}$, $0<\mu<1$, in a closed domain $\bar{D}$ in real or complex space if and only if each component is of class C^n on D and each of its n-th derivatives satisfies a uniform Hölder condition with exponent μ on D (and so can be extended to $\bar{D}$, etc.). If the tensor u is of class C^μ on D, D being in the real space, we define

$$h_\mu(u) = \sup |x_2 - x_1|^{-\mu} \|u(x_2) - u(x_1)\| \text{ for all } x_1, x_2 \text{ on } D \text{ with } x_1 \neq x_2;$$

$h^*_\mu(u)$ is defined correspondingly if D is in complex space. If u^k is a vector, $\nabla^p u^k$ denotes the tensor $u^k_{,\alpha_1 \cdots \alpha_p}$, the subscripts after the comma denoting patrial differentiation with respect to $x^{\alpha_1} \cdots x^{\alpha_p}$; the definition extends to tensors u. $B(x_0, R)$ denotes the ball $|x - x_0| < R$ in real ν-space and $B_R = B(0, R)$. B_{hR} denotes the set of all complex vectors $x = (x^1, \cdots, x^\nu) = x_1 + ix_2$ in which

$$B_{hR}: \ |x_1| < R, \ |x_2| < h(R - |x_1|).$$

If condition (1.4) holds for all x on a closed domain $\bar{D}$ and $L(x, \lambda)$ is continuous it follows that

$$m_1 > 0, \text{ where } m_1 = \min |L(x, \lambda)| \text{ for } x \in \bar{D}, |\lambda| = 1.$$

We let M denote a bound for the coefficients occurring in any of the L_{jk} operators used. The constants K_1, K_2, K_{3Q}, etc., and $Z_1, Z_2, \cdots$ will depend only on the numbers $\nu, N, s_1, \cdots, s_N$, $t_1, \cdots, t_N, m_1$, and M unless the numbers μ and h enter in which case we shall write $K_1(\mu, h)$, etc. The K's are permanent constants whereas the Z's are used only in the course of a single proof, so that Z_1 may denote different constants in different proofs.

We now outline our method of proof. Suppose we have a solution of (1.1) in which each u^k is of class $C^{t_k+\mu}$, $0<\mu<1$, near some point x_0. By a translation of axes, we may assume $x_0 = 0$. Next, let $P^k(x)$ be that polynomial of degree t_k such that

$$\nabla^p P^k(0) = \nabla^p u^k(0), \qquad 0 \leqq p \leqq t_k. \tag{1.6}$$

Making the substitution

$$u^k = P^k + v^k \tag{1.7}$$

in (1.1) we obtain new equations of the same type in which $\nabla^p v^k(0) = 0$

for $0 \leqq p \leqq t_k$. If we expand the new ϕ_j about the origin we may rewrite the new equations in the form

$$L_{jk}v^k = M_{jk}v^k + \psi_j(x, \cdots, v^k, \cdots, \nabla^{s_j+t_k}v^k, \cdots) \tag{1.8}$$

where now the $L_{jk}(D)$ are operators with constant coefficients containing the derivatives or order exactly $s_j + t_k$, the $M_{jk}(D)$ are operators with constant coefficients of order $< s_j + t_k$, and ψ_j is the remainder term. Since L_{jk} contains derivatives of v^k of order $s_j + t_k$ and $s_j \leqq 0$, we may differentiate (1.8) $(-s_j)$ times with respect to the x^α. Since all derivatives of v^k up to the order t_k vanish at the origin, we note that the Taylor expansion of ψ_j begins with a homogeneous polynomial in x of degree $1 - s_j$, linear terms in the $\nabla^p v^k$ with coefficients homogenous and linear in x, and quadratic terms in the $\nabla^p v^k$ with constant coefficients.

In Sections 2 and 3 we prove the following theorem:

THEOREM A. *For each $R > 0$, there exists a linear transformation $u = P_R(f)$ from the space of vectors f in which f_j is of class $C^{\mu-s_j}$, $0 < \mu < 1$, on $\bar{B}_R$ with $\nabla^p f_j(0) = 0$ for $0 \leqq p \leqq -s_j$ into the space of vectors u in which u^k is of class $C^{t_k+\mu}$ on $\bar{B}_R$ with $\nabla^p u^k(0) = 0$, $0 \leqq p \leqq t_k$ such that*

$$L_{jk}u^k = f_j \tag{1.9}$$

in which L_{jk} is an operator of the type in (1.8). *Moreover,*

$$\max_{1 \leqq k \leqq N} h_\mu(\nabla^{t_k} u^k) \leqq K_1(\mu) \max_{1 \leqq j \leqq N} h_\mu(\nabla^{-s_j} f_j). \tag{1.10}$$

In the space of vectors u described in this theorem, we introduce the norm

$$\| u \| = \max_{1 \leqq k \leqq N} h_\mu(\nabla^{t_k} u^k) \qquad (x \text{ on } \bar{B}_R). \tag{1.11}$$

The resulting space is clearly a Banach space which we denote by E_R. Clearly, if $u \in E_R$, we see that

$$\begin{aligned} &\| \nabla^{t_k} u^k(x) \| \leqq \| u \| \cdot | x |^\mu, \\ &\| \nabla^p u^k(x) \| \leqq \| u \| \cdot | x |^{t_k - p + \mu}/(1+\mu) \cdots (t_k - p + \mu), \\ &\qquad\qquad 0 \leqq p < t_k. \end{aligned} \tag{1.12}$$

We wish to replace (1.8) by an equation involving operators on E_R. To do this we write

$$v = V_R + H_R, \qquad V_R = P_R(f), \tag{1.13}$$

where f denotes the right side of (1.8). If v is given, as is assumed, V_R is

known and H_R is known. Moreover, H_R is a solution of the homogeneous equation (1.9) with $f=0$. We shall regard H_R as known and try to find an equation for V_R. To do this we substitute (1.13) into the right side of (1.8) and define

$$W_R = P_R\{M_{jk}H_R{}^k + \psi_j(x, \cdots, H_R, \cdots)\},$$

$$T_R(V_R; H_R) \tag{1.14}$$
$$= P_R\{M_{jk}V_R{}^k + \psi_j(x, \cdots, H_R + V_R, \cdots) - \psi_j(x, \cdots, H_R, \cdots)\}$$

Then (1.8) is equivalent to the operator equation

$$V_R - T_R(V_R; H_R) = W_R, \tag{1.15}$$

where W_R is, of course, known if H_R is.

Now, of course, $\| v \| \leqq M_1$ independently of R for our solution and so $\| H_R \| \leqq M_2$ for all $R \leqq R_0$, say. From (1.12), we see that if we restrict ourselves to elements H_R and V_R with norms $\leqq M_2$, then W_R and $T_R(V_R, H_R)$ are defined for sufficiently small R. By using (1.12), (1.14), the form of ψ_j, and that of the M_{jk} (operator of lower order than L_{jk}), it is straightforward to verify that

$$T_R(0; H_R) = 0, \quad \| T_R(V_{1R}; H_R) \tag{1.16}$$
$$- T_R(V_{2R}; H_R) \| \leqq \epsilon_1(R) \cdot \| V_{1R} - V_{2R} \|$$
$$\| W_R \| \leqq \epsilon_2(R), \quad \lim_{R \to 0} \epsilon_i(R) = 0, \| H_R \|, \| V_{1R} \|, \| V_{2R} \| \leqq M_2.$$

Thus if $R \leqq R_1$, say, there is a unique solution V_R of (1.15) for which $\| V_R \| \leqq M_2$ for any given H_R with $\| H_R \| \leqq M_2$.

For given numbers $R > 0$ and h, $0 < h < 1$, let us introduce the space $E^*{}_{hR}$ of vectors u in which u^k is analytic in B_{hR}, u^k is of class $C^{t_k+\mu}$ on $\bar{B}_{hR}$, and $\nabla^p u^k(0) = 0$ for $0 \leqq p \leqq t_k$ in which we define $\| u \|^*$ by (1.11) with h_μ replaced by h^*_μ. We introduce the corresponding space $F^*{}_{hR}$ of vectors f. Obviously, if $u \in E^*{}_{hR}$ for some $h > 0$, $u \in E_R$ and $\| u \|^* \geqq \| u \|$. In Section 4, the following two theorems are proved:

THEOREM B. *There is a number $h_0 = h_0(\mu, \nu, N, s_j, t_j, m_1, M) > 0$ such that if $f \in F^*{}_{hR}$ with $0 < h \leqq h_0$ and $u = P_R(f)$, then u can be extended to $\bar{B}_{hR}$ so that $u \in E^*{}_{hR}$ and*

$$\| u \|^* \leqq K_2(h, \mu) \max_{1 \leqq j \leqq N} h^*_\mu(\nabla^{-s_j} f_j). \tag{1.17}$$

THEOREM C. *If $H_R \in E_R$, then H_R can be extended analytically to any B_{hR} with $0 < h \leqq h_0$ and the extended function $\in E^*_{hR}$ with*

$$\| H_R \|^* \leqq K_3(h, \mu) \| H_R \|. \tag{1.18}$$

From Theorem B, it follows that *the operator $T_R(V_R, H_R)$ can be defined for functions V_R and H_R in any B_{hR} with $0 < h \leqq h_0$ and $0 < R \leqq R_2$ and, moreover, that* (1.16) *holds with the norm $\| \phi \|$ replaced by $\| \phi \|^*$* (and different functions $\epsilon_i(R)$).

Now, let us revert to our given solution v, and let V_R and H_R be defined as before. From Theorem C, we see that $\| H_R \|^* \leqq M_3$ for $0 < R \leqq R_3$. From the underlined results, we see that *if R is small enough, there are unique solutions V_R and V_{h_0R} in E_R and $E^*_{h_0R}$ respectively, of* (1.15) *with $\| V_R \| \leqq M_2$ and $\| \tilde{V}_{h_0R} \| \leqq M_3$ for our given H_R. Since $E^*_{h_0R} \subset E_R$, $\tilde{V}_{h_0R} = V_R$ so V_R and hence v are analytic; in fact they can be extended analytically to B_{h_0R} and their extensions $\in E^*_{h_0R}$.*

2. The particular solutions. In this section, we construct the particular solutions of equations (1.9) which were mentioned in the introduction and prove Theorem A. For the moment, we shall merely asume that the f_j are of class $C^{\mu - s_j}$ $(0 < \mu < 1)$ and shall drop the requirements that $\nabla^p f_j(0) = 0$ for $0 \leqq p \leqq -s_j$. These developments are very similar to those of Nirenberg and Douglis [3].

We define

$$G_{1n}(u) = (u^n/n!)(\log |u| - C_n), \tag{2.1}$$

$$C_n = 1 + 1/2 + \cdots + 1/n, \qquad G_{2n}(u) = u^n |u|/(n+1)!.$$

Then

$$G^{(k)}_{jn}(u) = G_{j,n-k}(u), \qquad 0 \leqq k \leqq n, j = 1, 2. \tag{2.2}$$

Next, if $m > 0$, we define

$$J^*(y) = \begin{cases} C_{1\nu} \displaystyle\int_{|\lambda|=1} G_{1m}(y \cdot \lambda) L^{-1}(\lambda) \, d\Sigma_\lambda, & \nu \text{ even}, \\ C_{2\nu} \displaystyle\int_{|\lambda|=1} G_{2m}(y \cdot \lambda) L^{-1}(\lambda) \, d\Sigma_\lambda, & \nu \text{ odd}, \end{cases} \tag{2.3}$$

$$J(y) = \begin{cases} \Delta^{\nu/2} J^*(y), & \nu \text{ even}, \\ \Delta^{(\nu+1)/2} J^*(y), & \nu \text{ odd}. \end{cases} \tag{2.4}$$

Let L be the scalar operator having $L(\lambda)$ as characteristic polynomial. Then

$$(2.5) \qquad LJ^*(y) = M_\nu(y) = \begin{cases} C_{1\nu} \int_{|\lambda|=1} G_{10}(y\cdot\lambda)\, d\Sigma_\lambda, & \nu \text{ even}, \\ C_{2\nu} \int_{|\lambda|=1} G_{20}(y\cdot\lambda)\, d\Sigma_\lambda, & \nu \text{ odd}. \end{cases}$$

Clearly J is homogeneous of degree $m-\nu$ unless ν is even and $m-\nu \geqq 0$ in which case

$$(2.6) \qquad J(y) = J_1(y)\log|y| + J_2(y),$$

where J_1 is a *polynomial* and J_1 and J_2 are homogeneous of degree $m-\nu$; and

$$(2.7) \qquad J(-y) = (-1)^m J(y).$$

Finally, for each j and a given $R>0$, we define

$$(2.8) \qquad F_j(x) = \int_{B_R} J(x-\xi) f_j(\xi)\, d\xi, \quad F^*_j(x) = \int_{B_R} J^*(x-\xi) f_j(\xi)\, d\xi.$$

The constants $C_{1\nu}$ and $C_{2\nu}$ are chosen so that

$$(2.9) \qquad \Delta^{-1+\nu/2} M(y) = \Gamma_0(y), \text{ if } \nu \text{ even}, \quad \Delta^{-1+(\nu+1)/2} M(y) = \Gamma_0(y) \text{ if } \nu \text{ odd},$$

$$\Gamma_0(y) = -(\nu-2)^{-1} |\Sigma|^{-1} \cdot |y|^{2-\nu}, \ \nu > 2, \ \Gamma_0 = \tfrac{1}{2\pi} \log|y|, \ \nu = 2$$

where $|\Sigma|$ is the $(\nu-1)$-measure of $\partial B(0,1)$; Γ_0 is just the elementary function for Laplace's equation.

It follows from (2.1)-(2.9) that

$$(2.10) \qquad LF^*_j(x) = \int_{B_R} M(x-\xi) f_j(\xi)\, d\xi.$$

From the form of M, it follows that we may differentiate the right side of (2.10) $(\nu-1)$ times under the integral sign. Thus from (2.9) and ordinary potential theory we see that

$$(2.11) \qquad \begin{cases} \Delta^{\nu/2} LF^*_j(x) = f_j(x), & \nu \text{ even}, \\ \Delta^{(\nu+1)/2} LF^*_j(x) = f_j(x), & \nu \text{ odd}. \end{cases}$$

In the next section, we shall see that each F_j is of class $C^{m-s_j+\mu}$ on $\bar{B}_R$ and that

$$(2.12) \qquad h_\mu(\nabla^{m-s_j} F_j) \leqq K_0 \cdot h_\mu(\nabla^{-s_j} f_j), \quad K_0 = K_0(\mu, \nu, N, m_1, M).$$

As soon as that is shown, it will follow that

$$(2.13) \qquad LF_j(x) = f_j(x), \quad x \in B_R \ (\bar{B}_R \text{ by continuity}).$$

In case $m=0$, then $L(\lambda)$ is just a constant, L is a constant operator, we

define $F_j(x) = L^{-1}f_j(x)$, and (2.13) holds, of course, and the differentiability properties of F_j hold.

Let us grant the statements above about the F_j. Let

$$\| L^{pj}(\lambda) \| = \| L_{jk}(\lambda) \|^{-1} \cdot L(\lambda) \tag{2.14}$$

and let $L^{pj}(D)$ be the corresponding operators; $L^{pj}(D) \equiv 0$ or is an operator of order $m - s_j - t_j \geqq 0$. Let us define

$$U^p(x) = L^{pj}F_j(x) \tag{2.15}$$

Then from (2.12) and (2.15), we see that U^p is of class $C^{t_p+\mu}$ on $\bar{B}_R$ so that

$$L_{jk}U^k(x) = L_{jk}L^{kl}F_l(x) = \delta^l{}_j LF_l(x) = LF_j(x) = f_j(x). \tag{2.16}$$

Finally, suppose

$$\nabla^p f_j(0) = 0, \qquad 0 \leqq p \leqq -s_l. \tag{2.17}$$

Let U be the solution (2.15) and let Q^k be the polynomial of degree t_k such that (2.18) holds, and define u^k by (2.18):

$$\nabla^p Q^k(0) = \nabla^p U^k(0),\ 0 \leqq p \leqq t_k,\ u^k(x) = U^k(x) - Q^k(x). \tag{2.18}$$

Then

$$L_{jk}u^k(x) = f_j(x) - q_j(x), \qquad j = 1, \cdots, N, \tag{2.19}$$

where q_j is a polynomial of degree $-s_j$. Differentiating (2.19) p times and setting $x = 0$, $p = 0, 1, \cdots, -s_j$, we see that $q_j(x) \equiv 0$. Thus Theorem A will be completely proved as soon as the differentiability properties of F_j are proved.

3. The analyticity of J and J^*; the differentiability of the F_j. We first prove the analyticity of J and J^*. To this end, we prove:

LEMMA 3.1. *Suppose y_0 is a real vector and $y_0 \neq 0$. Then there is a linear transformation*

$$\lambda_\alpha = \lambda_\alpha(\mu, y) \equiv h_\alpha{}^\beta(y)\mu_\beta \tag{3.1}$$

which is orthogonal for each real y near y_0, in which the $h_\alpha{}^\beta(y)$ are analytic and positively homogeneous of degree 0 for all complex y near y_0, and for which

$$\lambda_\alpha y^\alpha = \mu_1 |\, y \,|, \text{ where } |\, y \,| = [\sum_{\alpha=1}^{\nu} (y^\alpha)^2]^{\frac{1}{2}}. \tag{3.2}$$

Proof. Using a fixed orthogonal transformation, we easily reduce the problem to the case where

$$y_0{}^\alpha = |\, y_0 \,| \, \delta^\alpha{}_1. \tag{3.3}$$

Since we may solve (3.1) for μ, obtaining $\mu_\alpha = h_\beta{}^\alpha \lambda_\beta$ we see that (3.2) implies that $h_\alpha{}^1(y) = y^\alpha / |\, y \,|$. It is easy to see that there is a unique way to complete the matrix h to be orthogonal with all the minors in the upper left hand corner positive, and with $h_\alpha{}^\beta(y) = 0$ for $\alpha > \beta$; this is all for real y sufficiently near y_0. The $h_\alpha{}^\beta$ are analytic for complex y near y_0.

LEMMA 3.2. *$J^*(y)$ and $J(y)$ are analytic for all real $y \neq 0$.*

Proof. Suppose y_0 is real and $y_0 \neq 0$. From (2.3) we have

$$\begin{aligned} J^*(y) &= C_{1\nu} \int_{|\lambda|=1} G_m(\lambda \cdot y) L^{-1}(\lambda)\, d\Sigma_\lambda = J_1{}^*(y) \log |\, y \,| \\ &+ C_{1\nu} |\, y \,|^m \int_{|\mu|=1} ((\mu^1)^m / m!)[\log |\, \mu^1 \,| - C_m] L^{-1}[\lambda(\mu, y)]\, d\Sigma_\mu \end{aligned} \tag{3.4}$$

for all real y near y_0 if ν is even, $J_1{}^*$ being a polynomial of degree m. The analyticity of J^* follows in this case from (3.4) and that of J follows from (2.4). The proof for ν odd is similar.

ASSUMPTION. *We assume hereafter that $m > 0$ since the differentiability properties of the F_j are proved in the case $m = 0$.*

LEMMA 3.3. *The functions F_j of (2.8) are of class $C^{m+\mu-s_j-1}$ on $\bar{B}_R$ and each component of $\nabla^{m-s_j-1} F_j$ is of the form*

$$\begin{aligned} F_{j,\alpha_1 \cdots \alpha_{m-1}\beta_1 \cdots \beta_n}(x) &= -\int_{\partial B_R} \Gamma_{,\beta_2 \cdots \beta_n}(x-\xi) f_j(\xi)\, d\xi'_{\beta_1} \\ - \int_{\partial B_R} \Gamma_{,\beta_3 \cdots \beta_n}(x-\xi) f_{j,\beta_1}(\xi)\, d\xi'_{\beta_2} &- \cdots - \int_{\partial B_R} \Gamma(x-\xi) f_{j,\beta_1 \cdots \beta_{n-1}}(\xi)\, d\xi'_{\beta_n} \\ &+ \int_{B_R} \Gamma(x-\xi) f_{j,\beta_1 \cdots \beta_n}(\xi)\, d\xi, \quad n = -s_j; \end{aligned} \tag{3.5}$$

$$\Gamma(y) = J_{,\alpha_1 \cdots \alpha_{m-1}}(y); \tag{3.6}$$

the function Γ is homogeneous of degree $1-\nu$, analytic for $y \neq 0$, and

$$\Gamma(-y) = -\Gamma(y) \tag{3.7}$$

Proof. If $-s_j = n \geqq 1$, we see that

$$F_{j,\beta_1}(x) = \int_{B_R} J_{,\beta_1}(x-\xi) f_j(\xi)\, d\xi = -\int_{B_R} [\partial J(x-\xi)/\partial \xi^{\beta_1}] f_j(\xi)\, d\xi$$
$$= -\int_{\partial B_R} J(x-\xi) f_j(\xi)\, d\xi'_{\beta_1} + \int_{B_R} J(x-\xi) f_{j,\beta_1}(\xi)\ d\xi.$$

The form (3.5) follows by repeating this argument and then differentiating with respect to $x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}$. That Γ is analytic for $y \neq 0$ and satisfies (3.7) follows from (3.6), Lemma 3.2, and (2.7).

From Lemma 3.3, it follows that the desired differentiability results for the F_j will be proved if we can show that if

$$U(x) = \int_{B_R} \Gamma(x-\xi) f(\xi)\, d\xi, \quad f \in C^{0+\mu} \text{ on } \bar{B}_R, \tag{3.8}$$

$$W_\gamma(x) = -\int_{\partial B_R} \Gamma(x-\xi) f(\xi)\, d\xi'_\gamma \quad f \in C^{q+\mu} \text{ on } \bar{B}_R, q \geqq 0, \tag{3.9}$$

then U is of class $C^{1+\mu}$ on $\bar{B}_R$ and W is of class $C^{q+\mu}$ on $\bar{B}_R$, and

$$h_\mu(\nabla U) \leqq K_4(\mu) h_\mu(f), \qquad h_\mu(\nabla^q W) \leqq K_{5q}(\mu) h_\mu(\nabla^{q+1} f) \tag{3.10}$$

ASSUMPTION. *In all the following lemmas,* Γ *is assumed to have the properties given in Lemma* 3.3.

To prove the statements above, we first prove the following well-known lemma which is hard to find in the literature (but cf. [6]).

LEMMA 3.4. *Suppose* Γ *satisfies the conditions of Lemma* 3.3. *Then*

$$\int_{\partial B(x_0,R)} \Gamma(\xi-x)\, d\xi'_\gamma = \int_{\partial B(0,1)} \Gamma(\eta)\, d\eta'_\gamma \equiv \int_{\partial B(0,1)} \Gamma(\eta)\eta^\gamma\, dS_\eta, \quad x \in B(x_0, R). \tag{3.11}$$

Proof. If $x = x_0$, the result is obvious from the homogeneity. If $x \in B(x_0, R)$ but $x \neq x_0$, and if $|\eta| = 1$, we define ξ and λ by

$$\xi - x_0 = (x - x_0) + \lambda\eta, \quad R^2 = r^2 + 2\lambda r(\omega \cdot \eta) + \lambda^2,$$
$$\omega = (x - x_0)/|x - x_0|, \quad r = |x - x_0|, \tag{3.12}$$
$$\lambda = -r\cos\psi + (R^2 - r^2\sin^2\psi)^{\frac{1}{2}}, \cos\psi = \omega \cdot \eta.$$

Then

$$|\xi - x| = \lambda, \quad d\xi'_\gamma = [(\xi^\gamma - x_0^\gamma)/R]\, dS_\xi = [(\xi^\gamma - x_0^\gamma)/R] \cdot \lambda^{p-1} \sec X\, dS_\eta, \tag{3.13}$$

where X is the angle between the vectors $\vec{x\xi}$ and $\vec{x_0\xi}$, and

$$\sec X = 2R\lambda/(R^2 + \lambda^2 - r^2) = R/(R^2 - r^2\sin^2\psi)^{\frac{1}{2}} \tag{3.14}$$

from (3.12). Substituting (3.13) and (3.14) in (3.11) and using the properties of Γ, we obtain

$$\int_{\partial B(x_0,R)} \Gamma(\xi - x)d\xi'_\gamma = \int_{\partial B(0,1)} \Gamma(\eta)\cdot(\xi^\gamma - x_0{}^\gamma)/R\cdot R/(R^2 - r^2\sin^2\psi)^{\frac{1}{2}}dS_\eta$$
$$= \int_{\partial B(0,1)} \Gamma(\eta)\eta^\gamma\, dS_\eta$$
$$+ \int_{\partial B(0,1)} \Gamma(\eta)\cdot[(x^\gamma - x_0{}^\gamma) + r(\omega\cdot\eta)\eta^\gamma]/(R^2 - r^2\sin^2\psi)^{\frac{1}{2}}\, dS_\eta$$

and the last integral is zero on account of (3.7).

As an immediate consequence of Lemma 3.4, we obtain

LEMMA 3.5. *There are constants* $C^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n\gamma}$ *which depend on* Γ *but whose magnitudes depend only on* ν, N, s_j, t_j, m_1, *and* M *such that*

$$-\int_{\partial B(x_0,R)} \Gamma_{,\alpha_1\cdots\alpha_n}(x-\xi)\cdot(\xi^{\beta_1} - x^{\beta_1})\cdot\cdots\cdot(\xi^{\beta_n} - x^{\beta_n})\,d\xi'_\gamma$$
$$= C^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n\gamma},\ x\in B(x_0,R),$$

$$-\int_{\partial B(x_0,R)} \Gamma_{,\alpha_1\cdots\alpha_n}(x-\xi)\cdot(\xi^{\beta_1} - x^{\beta_1})\cdot\cdots\cdot(\xi^{\beta_p} - x^{\beta_p})\,d\xi'_\gamma = 0,\ p<n,$$
$$x\in B(x_0,R),$$

$$\int_{\partial B(x_0,R)-B(x_1,\rho)} \Gamma_{,\alpha_1\cdots\alpha_n}(x-\xi)\cdot(\xi^{\beta_1} - x^{\beta_1})\cdot\cdots\cdot(\xi^{\beta_p} - x^{\beta_p})\,d\xi = 0,\ p<n,$$
$$x\in B(x_1,\rho)\subset B(x_0,R).$$

LEMMA 3.6. *Suppose* W_γ *is given by* (3.9), *where* $f\in C^{q+\mu}$ *on* $\bar{B}_R$. *Then* $W\in C^{q+\mu}$ *on* $\bar{B}_R$ *and* $h_\mu(\nabla^q W_\gamma)\leqq K_{5q}(\mu)h_\mu(\nabla^q f)$.

Proof. Differentiating (3.9), $q+1$ times and using Lemma 3.5, we obtain

$$\nabla^{q+1}W = -\int_{\partial B_R} \nabla^{q+1}\Gamma(x-\xi)[f(\xi) - \sum_{j=0}^{q}\nabla^j f(x)\cdot(\xi-x)^j/j!]\,d\xi'_\gamma,\ x\in B_R \tag{3.15}$$

where the sum in the bracket denotes the Taylor expansion of f out to the terms of order q. Taking magnitudes, we obtain

$$|\nabla^{q+1}W_\gamma(x)|\leqq\Gamma_{q+1}\cdot h_\mu(\nabla^q f)\int_{\partial B_R}|x-\xi|^{\mu-\nu}\,dS_\xi\leqq Z_1(\mu)h_\mu(\nabla^q f)\cdot(R-|x|)^{\mu-1}$$

Since any two points in B_R can be joined by an arc $x=x(s)$, $0\leqq s\leqq l$ (s arc length) such that

$$\int_0^l (R-|x(s)|)^{\mu-1}\,ds\leqq 3\mu^{-1}|x_1-x_2|^\mu,$$

the result follows.

LEMMA 3.7. *Suppose U is given by* (3.8), *where $f \in C^\mu$ on $\bar{B}_R$. Then U is of class $C^{1+\mu}$ on B_R and*

$$U_{,\alpha}(x) = C_\alpha f(x) + \int_{B_R} \Gamma_\alpha(x-\xi)[f(\xi)-f(x)]\,d\xi \tag{3.16}$$

$$x \in B_R, C_\alpha = -\int_{\partial B_R} \Gamma(y)\,dy'_\alpha.$$

Moreover, $h_\mu(\nabla U) \leqq K_4(\mu) h_\mu(f)$.

Proof. Let us extend f to be of class C^μ on $\bar{B}_{2R}$ with the same Hölder condition. Then

$$U = U_1 - U_2, \qquad U_1(x) = \int_{B_{2R}} \Gamma(x-\xi) f(\xi)\,d\xi, \tag{3.17}$$

$$U_2 = \int_{B_{2R}-B_R} \Gamma(x-\xi) f(\xi)\,d\xi.$$

We have

$$U_{2,\alpha}(x) = \int_{B_{2R}-B_R} \Gamma_{,\alpha}(x-\xi) f(\xi)\,d\xi$$

$$= \int_{B_{2R}-B_R} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]\,d\xi, x \in B_R \tag{3.18}$$

$$U_{2,\alpha\beta}(x) = \int_{B_{2R}-B_R} \Gamma_{,\alpha\beta}(x-\xi)[f(\xi)-f(x)]\,d\xi$$

using Lemma 3.5 and the fact that $\Gamma_{,\alpha}(x-\xi) = -\partial\Gamma(x-\xi)/\partial\xi^\alpha$. From the second equation in (3.18) we see that

$$|\nabla^2 U_2(x)| \leqq Z_1(\mu) h_\mu(f)(R-|x|)^{\mu-1} \tag{3.19}$$

As in the proof of Lemma 3.6, it follows from (3.19) that

$$h_\mu(\nabla U_2) \leqq Z_2(\mu) h_\mu(f). \tag{3.20}$$

We let

$$U_{1\rho}(x) = \int_{B_{2R}-B(x,\rho)} \Gamma(x-\xi) f(\xi)\,d\xi, x \in B_R, 0 < \rho < R. \tag{3.21}$$

Then for such x and ρ,

$$U_{1\rho,\alpha}(x) = U_{3\rho\alpha}(x) + U_{4\rho\alpha}(x)$$

$$U_{3\rho\alpha}(x) = -\int_{\partial B(x,\rho)} \Gamma(x-\xi)[f(\xi)-f(x)]\,d\xi'_\alpha + C_\alpha f(x),$$

$$C_\alpha = -\int_{\partial B_1} \Gamma(y)\,dy'_\alpha$$

$$U_{4\rho\alpha}(x) = \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha}(x-\xi) f(\xi)\,d\xi = \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]\,d\xi.$$

Clearly, as $\rho \to 0$, $U_{3\rho\alpha}(x)$ and $U_{4\rho\alpha}(x)$ converge uniformly on $\bar{B}_R$ to

$$U_{3\alpha}(x) = C_\alpha f(x),\ U_{4\alpha} = \int_{B_{2R}} \Gamma_\alpha(x-\xi)[f(\xi) - f(x)]\,d\xi. \tag{3.22}$$

In fact,

$$|U_{4\rho\alpha}(x) - U_{4\alpha}(x)| \leqq Z_3(\mu) h_\mu(f) \cdot \rho^\mu,\ Z_3 = Z_3(\mu, \nu, N, s, t, m_1, M). \tag{3.23}$$

But now, if $x, x_1, x_2, \bar{B}_R$,

$$\begin{aligned} U_{4\rho\alpha,\beta}(x) &= -\int_{\partial B(x,\rho)} \Gamma_{,\alpha}(x-\xi)[f(\xi) - f(x)]\,d\xi'_\beta \\ &\quad + \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha\beta}(x-\xi)[f(\xi) - f(x)]\,d\xi \end{aligned} \tag{3.24}$$

$$|\nabla U_{4\rho\alpha}(x)| \leqq Z_4(\mu) h_\mu(f) \cdot \rho^{\mu-1},\ |U_{4\rho\alpha}(x_2) - U_{4\rho\alpha}(x_1)| \leqq Z_4(\mu) \cdot h_\mu(f) \cdot \rho^\mu,\ |x_1 - x_2| = \rho.$$

Assuming $x_1, x_2 \in \bar{B}_R$ and setting $|x_1 - x_2| = \rho$, we see from (3.22) and (3.23) that

$$h_\mu(U_{4\alpha}) \leqq Z_5(\mu) \cdot h_\mu(f), h_\mu(U_{3\alpha}) \leqq Z_6(\mu) \cdot h_\mu(f). \tag{3.25}$$

The lemma follows.

4. Proofs of Theorems B and C. In this section we prove Theorems B and C as stated in the introduction. We observe first:

LEMMA 4.1. *There is an $h_0(\nu, N, s_j, t_j, m_1, M) > 0$ such that, if $0 < h \leqq h_0$, then $J(y)$ is analytic for all complex $y = y_1 + iy_2 \neq 0$ for which $|y_2| \leqq h|y_1|$; J satisfies (2.7) and, either*

$$J(ky) = k^{m-\nu} J(y),\ k \text{ real},\ k > 0, \tag{4.1}$$

or else J has the form (2.6) where J_1 is a polynomial of degree $m - \nu (\geqq 0)$ and J_2 satisfies (4.1).

This follows directly from the homogeneity properties of J and Lemma 3.2.

GENERAL ASSUMPTION. *We shall assume that h is a fixed positive number such that $0 < h \leqq h_0$.*

We wish to show that if the $f_j(x)$ are analytic for $x \in B_{hR}$ and $f_j \in C^{\mu-s_j}$ on $\bar{B}_{hR}$, then the $F_j(x)$ as defined in (2.8) are analytic in B_{hR} and $F_j \in C^{m-s_j+\mu}$ on $\bar{B}_R$.

Remark. If $m=0$, these statements about the F_j are obvious, (2.13) holds and Theorem B follows. In this case, if u is a solution of (1.9) with $f_j=0$, we see that each term on the left in (1.9) is of class $C^{\mu-s_j}$, and hence, the operators L^{pj} may be applied to yield

$$Lu^p=0 \text{ or } u^p=0$$

since L is a constant $\neq 0$ and those $L^{pj}\neq 0$ are of order $-s_j-t_p\geqq 0$, Thus Theorem C holds. *Accordingly we assume $m>0$ for the remainder of this section.*

Now, we shall extend the functions $F_j(x)$ to the domains B_{hR} by a device used by E. E. Levi [5] and E. Hopf [4]. Let $x=x_1+ix_2\in B_{hR}$. For such x, define d_x as the distance of x from ∂B_{hR} (in 2ν-space) and ρ_x as the sup. of numbers $\rho>0$ such that all points x_1+s+ix_2 with $|s|<\rho$ belong to B_{hR}. Then, clearly

$$\rho_x=[(1+h^2)^{\frac{1}{2}}/h]d_x. \tag{4.2}$$

For such x, we define the ν-dimensional cone $S(x)$ by

$$\begin{aligned} &S(x):\ \xi^\alpha=s^\alpha+i(1-\lambda)x_2^\alpha,\ \lambda=\lambda(s,x_1),\ 0\leqq\lambda\leqq 1, \\ &(R^2-|x_1|^2)\lambda^2-2\lambda x_1\cdot(s-x_1)-|s-x_1|^2=0,\ s\in B_R\ (s \text{ real}). \end{aligned} \tag{4.3}$$

We note that $S(x)$ has a 1-1 projection on B_R, that λ is the ratio $|s-x_1|/|S-x_1|$ where S is the intersection of the ray x_1s with ∂B_R, that λ is real-analytic in (s,x_1) for $s\in B_R$, $x\in B_R$, $x\neq s$, and that the derivatives of λ are uniformly bounded if $s\neq x_1$, $x_1\in B_{R'}$, $R'<R$. Also, we note that

$$\begin{aligned} \lambda|x_2| &= (|s-x_1|/|S-x_1|)\,|x_2| \\ &\leqq (|s-x_1|/|S-x_1|)\cdot h\cdot(R-|x_1|)\leqq h|x_1-s| \\ (1-\lambda)|x_2| &\leqq (1-\lambda)h(R-|x_1|)\leqq h(R-|s|), \\ s &= (1-\lambda)x_1+\lambda S,\ S\in\partial B_R. \end{aligned} \tag{4.4}$$

The second of these follows since the maximum of $|(1-\lambda)x_1+\lambda S|$ for all S on ∂B_R and a given λ, $0<\lambda<1$, is taken on for S the nearest point to x_1. Thus, for any $s\neq x_1$ with $s\in B_R$, we see that ξ (as defined in (4.3)) $\in B_{hR}$ and that $[x_1-s+i\lambda x_2]$ belongs to the set where J is analytic. Thus, we may define

$$\begin{aligned} F_j(x) &= \int_{S(x)} J(x-\xi)f_j(\xi)\,d\xi \\ &\equiv \int_{B_R} J[(x_1-s)+i\lambda x_2]\cdot f_j[s+i(1-\lambda)x_2]\,j(s,x)\,ds, \end{aligned} \tag{4.5}$$

where $j(s,x)=\partial(\xi^1,\cdots,\xi^\nu)/\partial(s^1,\cdots,s^\nu)=1-ix_2{}^\gamma\lambda_{s^\gamma}$, $x\in B_{hR}$.

Next, let us define

$$\lambda_\rho(s;x_1)=\lambda(s,x_1)[\,|s-x_1|-\rho]/[\,|s-x_1|-\rho\lambda(s,x_1)\,],\ x_1,s\in B_R,\ s\neq x_1,\quad 0<\rho<R-|x_1|.$$

We see that λ_ρ is real analytic for these values of (s,x_1), $\lambda_\rho=0$ for $|s-x_1|=\rho$, $\lambda_\rho=1$ for s on ∂B_R, and the derivatives of λ_ρ are uniformly bounded for $x_1\in B_{R'}$, $R'<R$, where they converge uniformly to those of λ as $\rho\to 0$. Finally we see, as above, that

$$(4.6)\qquad \lambda_\rho|x_2|\leqq h|x_1-s|,\ (1-\lambda_\rho)|x_2|\leqq h(R-|s|),\quad s\in\bar B_R-B(x_1,\rho),\qquad 0<\rho<\rho_x.$$

Accordingly, if we define $S_\rho(x)$ by

$$(4.7)\qquad S_\rho(x):\ \xi^\alpha=s^\alpha+i(1-\lambda_\rho)x_2{}^\alpha,\ s\in\bar B_R-B(x_1,\rho),\qquad 0<\rho<\rho_x,$$

then the points of $S_\rho(x)\in B_{hR}$ if x does. Thus, if f_j is analytic in B_{hR}, we may define

$$(4.8)\qquad F_{\rho j}(x)=\int_{S_\rho(x)}J(x-\xi)f_j(\xi)\,d\xi \equiv\int_{B_R-B(x_1,\rho)}J[(x_1-s)+i\lambda_\rho x_2]f_j[s+i(1-\lambda_\rho)x_2]j_\rho(s,x_1)ds,\quad x\in B_{hR},\ \rho_x>\rho.$$

LEMMA 4.2. *Suppose $x\in B_{hR}$, $0<\rho<\rho_x$, and ϕ is analytic in a neighborhood of $S_\rho(x)$. Then*

$$\int_{S_\rho(x)}\phi_{,\alpha}(\xi)\,d\xi=\int_{\partial B_R}\phi(s)\,ds'_\alpha-\int_{\partial B(x_1,\rho)}\phi(s+ix_2)\,ds'_\alpha.$$

Proof. For the reader may verify that

$$\phi_{,\alpha}[s+i(1-\lambda_\rho)x_2]\cdot j_\rho(s,x_1)=(\partial/\partial s^\alpha)\phi[s+i(1-\lambda_\rho)x_2]-ix_2{}^\beta\{(\partial/\partial s^\alpha)(\phi\lambda_{\rho,s^\beta})-(\partial/\partial s^\beta)(\phi\lambda_{\rho,s^\alpha})\},$$

$$\lambda_{\rho,s^\beta}ds'_\alpha-\lambda_{\rho,s^\alpha}ds'_\beta=0,\ s\in\partial B_R,\ s\in\partial B(x_1,\rho).$$

LEMMA 4.3. *If the f_j are analytic in B_{hR} and F_j and $F_{\rho j}$ are defined as in (4.5) and (4.8), then F_j and $F_{\rho j}$ are analytic on their respective domains of definition and $F_{\rho j}(x)$ converges uniformly to $F_j(x)$ on any compact subset of B_{hR} as $\rho\to 0$. Moreover, the components of $\nabla^{m-s_j-1}F_j$ are given by (3.5), except that the last integral over the ball B_R is to be replaced by*

$$\int_{S(x)} \Gamma(x-\xi) f_{j,\beta_1 \cdots \beta_n}(\xi)\, d\xi \qquad (n = -s_j), \tag{4.9}$$

the integral in (4.9) *being given by* (4.5) *with* J *replaced by* Γ. *Moreover,* Γ *satisfies the conditions of Lemma* 3.3 *for the complex* y *in Lemma* 4.1.

Proof. Since $m > 0$, we see from Lemma 4.1 and the discussion of λ and λ_ρ that the $F_{\rho j}(x)$ converge uniformly to F_j on any compact subset of B_{hR}. Moreover, if $m > 1$, the derivatives of $F_{\rho j}$ with respect to ${x_1}^\alpha$ and ${x_2}^\alpha$ converge similarly to those of F. We wish to show that F_j and $F_{\rho j}$ are analytic on B_{hR}. It is clear that we may approximate f_j and its derivatives of order $\leqq -s_j$ uniformly by functions $f_{jn}(x) = f_j[(1-n^{-1})x]$ which are analytic on $\bar{B}_{hR}$.

Differentiating (4.8), we see that (if $m > 1$)

$$\begin{aligned} F_{\rho j,{x_2}^\alpha}(x_1 + ix_2) &= \int_{B_R - B(x_1,\rho)} iJ_{,\alpha} \cdot f_j \cdot j_\rho \, ds \\ &\quad + \int_{B_R - B(x_1,\rho)} i\{(1-\lambda_\rho) \cdot j_\rho \cdot (Jf_{j,\alpha} - J_{,\alpha} f_j) - \lambda_{\rho s^\alpha} J f_j\}\, ds \end{aligned} \tag{4.10}$$

$$\begin{aligned} F_{\rho j,{x_1}^\alpha}(x_1 + ix_2) &= -\int_{\partial B(x_1,\rho)} Jf_j j_\rho \, ds'_\alpha + \int_{B_R - B(x_1,\rho)} \Gamma_{,\alpha} f_j j_\rho \, ds \\ &\quad + \int_{B_R - B(x_1,\rho)} -ix_2{}^\beta \{j_\rho \cdot \lambda_{\rho,{x_1}^\alpha}(Jf_{j,\beta} - J_{,\beta} f_j) + Jf_j \lambda_{\rho,s^\beta {x_1}^\alpha}\}\, ds, \end{aligned} \tag{4.11}$$

where the arguments in the functions are those in (4.8); here we have assumed that the f_j are analytic on $\bar{B}_{hR}$. The reader may verify that

$$\begin{aligned} -ix_2{}^\beta \{j_\rho \cdot \lambda_{\rho,{x_1}^\alpha} \cdot (Jf_{j,\beta} - J_{,\beta} f_j) + Jf_j \lambda_{\rho,s^\beta {x_1}^\alpha}\} &= (\partial/\partial s^\beta)(-ix_2{}^\beta J f_j \lambda_{\rho,{x_1}^\alpha}) \\ i\{(1-\lambda_\rho) j_\rho (Jf_{j,\alpha} - J_{,\alpha} f_j) - \lambda_{\rho,s^\alpha} J f_j\} &= (\partial/\partial s^\alpha)[i(1-\lambda_\rho) J f_j] \\ &\quad + (\partial/\partial s^\alpha)[x_2{}^\beta (1-\lambda_\rho)\lambda_{\rho,s^\beta} J f_j] - (\partial/\partial s^\beta)[x_2{}^\beta (1-\lambda_\rho)\lambda_{\rho,s^\alpha} J f_j]. \end{aligned} \tag{4.12}$$

Using the facts that $\lambda_\rho(s, x_1) = 0$ on $\partial B(x_1, \rho)$, $\lambda_\rho(s, x_1) = 1$ for s on ∂B_R, so that $\lambda_{\rho,{x_1}^\alpha} = 0$ for s on ∂B_R, and the relations (4.10)-(4.12), we obtain

$$\begin{aligned} F_{\rho j,{x_2}^\alpha}(x_1 + ix_2) &= \int_{B_R - B(x_1,\rho)} iJ_{,\alpha} f_j j_\rho \, ds - i \int_{\partial B(x_1,\rho)} Jf_j \, ds'_\alpha \\ F_{\rho j,{x_1}^\alpha}(x_1 + ix_2) &= \int_{B_R - B(x_1,\rho)} J_{,\alpha} f_j j_\rho \, ds - \int_{\partial B(x_1,\rho)} Jf_j \, ds'_\alpha. \end{aligned} \tag{4.13}$$

In obtaining these relations, we used the facts, implied by the fact that $\lambda_\rho(s, x_1) = 0$ for all $x_1 \in B_{R-\rho}$ and all s on $\partial B(x_1, \rho)$, that

$$\lambda_{\rho,x_1^\alpha}(s,x_1) = -\lambda_{\rho,s^\alpha}(s,x_1), \quad \lambda_{\rho,s^\alpha}\,ds'_\beta - \lambda_{\rho,s^\beta}\,ds'_\alpha = 0, \quad s \in \partial B(x_1,\rho). \tag{4.14}$$

The result (4.13) shows that the $F_{\rho j}$ are analytic and that

$$F_{\rho j,\alpha}(x) = \int_{S_\rho(x)} J_{,\alpha}(x-\xi) f_j(\xi)\,d\xi - \int_{\partial B(x_1,\rho)} J(x_1-s) f_j(s+ix_2)\,ds'_\alpha, \tag{4.15}$$

all provided that the f_j are analytic on $\bar{B}_{hR}$.

In case the f_j are analytic on $\bar{B}_{hR}$, we wish to show that

$$F_{\rho j,\alpha}(x) = -\int_{\partial B_R} J(x-s) f_j(s)\,ds'_\alpha + \int_{S_\rho(x)} J(x-\xi) f_{j,\alpha}(\xi)\,d\xi. \tag{4.16}$$

But (4.16) follows from (4.15) and Lemma 4.2. The restriction that f_j be be analytic on $\bar{B}_{hR}$ is easily replaced by the restrictions as stated. As in Lemma 3.3, the process may be repeated. Finally we may let $\rho \to 0$.

As in Section 3, our desired results concerning the F_j will follow from our study of the functions U and W_γ defined by

$$U(x) = \int_{S(x)} \Gamma(x-\xi) f(\xi)\,d\xi, \quad f \in C^{0+\mu} \text{ on } \bar{B}_{hR}, \quad x \in B_{hR} \tag{4.17}$$

$$W_\gamma(x) = -\int_{\partial B_R} \Gamma(x-s) f(s)\,ds'_\gamma, \quad f \in C^{q+\mu} \text{ on } \bar{B}_{hR}, \tag{4.18}$$

where f is analytic on B_{hR} in both vases and Γ satisfies

Condition A. $\Gamma(y)$ is analytic for all complex $y = y_1 + iy_2 \neq 0$ in which $|y_2| \leqq h|y_1|$ and Γ satisfies (3.7) and

$$\Gamma(ky) = k^{1-\nu}\Gamma(y), \quad k \text{ real}, \quad k > 0, \tag{4.19}$$

for such y.

LEMMA 4.4. *The first two results of Lemma* 3.5 *hold for* $x \in B_h(x_0,R)$, *the set of all* $x = x_1 + ix_2$ *such that* $|x_2 - x_{02}| \leqq h(R - |x_1 - x_{01}|)$, $x_0 = x_{01} + ix_{02}$. *The third result is replaced by*

$$\int_{S_\rho(x)} \Gamma_{,\alpha_1\cdots\alpha_n}(x-\xi)\cdot(\xi^{\beta_1} - x^{\beta_1})\cdots(\xi^{\beta_p} - x^{\beta_p})\,d\xi = 0, \quad p < n, \quad x \in B_{hR}.$$

LEMMA 4.5. *Suppose* Γ *satisfies Condition* A, $f \in C^{q+\mu}$ *for real* x *on* $\bar{B}_{hR}$ *and* W_γ *is defined by* (4.18). *Then* $W_\gamma(x)$ *is analytic on* B_{hR} *and* $W_\gamma \in C^{q+\mu}$ *on* $\bar{B}_{hR}$ *and*

$$h^*_\mu(\nabla^q W_\gamma) \leqq K_{6q}(h,\mu)\cdot h_\mu(\nabla^q f).$$

Proof. It is obvious that W_γ is analytic on B_{hR}. Differentiating $q+1$ times and using Lemma 4.4, we obtain

$$\nabla^{q+1}W_\gamma(x) = -\int_{\partial B_R} \nabla^{q+1}\Gamma(x-\xi)\,[f(\xi) - \sum_{j=0}^{q} \nabla^j f(x_1)\cdot(\xi-x_1)^j/j!]\,d\xi'_\gamma,$$
$$x = x_1 + ix_2 \in B_{hR}.$$

From Condition A, we see that

$$|\nabla^{q+1}\Gamma(y)| \leqq Z_1(h)\cdot\|y\|^{-\nu-q} \text{ for } |y_2| \leqq h|y_1|,\ y \neq 0.$$

Since $|x_2| \leqq h(R-|x_1|)$, we see that

$$|\xi - x_1| \leqq \|\xi - x\| = [|\xi-x_1|^2 + |x_2|^2]^{\frac{1}{2}} \leqq (1+h^2)^{\frac{1}{2}}|\xi - x_1|.$$

The remainder of the proof of Lemma 3.6 may now be carried over.

LEMMA 4.6. *If f is analytic on B_{hR} and $f \in C^\mu$ on $\bar{B}_{hR}$, then*

$$|\nabla f(x)| \leqq K_7 h^*_\mu(f)\cdot d_x^{\mu-1},\ |\nabla^2 f(x)| \leqq K_8 h^*_\mu(f)\cdot d_x^{\mu-2}, x \in B_{hR},$$

where K_7 and K_8 depend only on ν and h.

Proof. Clearly there is a constant κ, $0<\kappa<1$, which depends only on ν and h that the polycylinder $|\xi^\alpha - x^\alpha| < \kappa d_x$, $\alpha = 1,\cdots,\nu$, lies in B_{hR}. Since $f \in C^\mu$, we have

$$f(x) = (2\pi i)^{-\nu}\int_\gamma \cdots \int_\gamma f(\xi)\,d\xi^1\cdots d\xi^\nu/(\xi^1-x^1)\cdots(\xi^\nu - x^\nu),$$
$$(\gamma : |\xi - x| = \kappa d_x)$$

$$\bar{\zeta}^\alpha f_{,\alpha}(x) = (2\pi i)^{-\nu}\int_\gamma \cdots$$
$$\int_\gamma \bar{\zeta}^\alpha[f(\xi) - f(x)]\,d\xi^1\cdots d\xi^\nu/[(\xi^1-x^1)\cdots(\xi^\nu - x^\nu)(\xi^\alpha - x^\alpha)]$$

for any complex ζ^α. The result follows from the fact that

$$|(\xi^\alpha - x^\alpha)^{-1}\bar{\zeta}^\alpha| \leqq \|\zeta\|\cdot(\kappa d_x)^{-1}.$$

The second result is proved similarly.

LEMMA 4.7. *If f satisfies the conditions of Lemma 4.6 and U is defined by (4.17), then U is analytic on B_{hR} and $U \in C^{1+\mu}$ on $\bar{B}_{hR}$ with*

$$U_{,\alpha}(x) = C_\alpha f(x) + \int_{S(x)} \Gamma_{,\alpha}(x-\xi)\,[f(\xi) - f(x)]\,d\xi, \tag{4.20}$$

$$h^*_\mu(\nabla U) \leqq K_9(h,\mu)h^*_\mu(f). \tag{4.21}$$

Proof. Define $U_\rho(x) = \int_{S_\rho(x)} \Gamma(x-\xi) f(\xi)\, d\xi$. Then, we may proceed as in the proof of Lemma 4.3 to obtain

$$
\begin{aligned}
U_{\rho,\alpha}(x) &= \int_{S_\rho(x)} \Gamma_{,\alpha}(x-\xi) f(\xi) d\xi - \int_{\partial B(x_1,\rho)} \Gamma(x_1-s) f(s+ix_2) ds'_\alpha \\
&= \int_{S_\rho(x)} \Gamma_{,\alpha}(x-\xi)\,[f(\xi)-f(x)]\, d\xi + C_\alpha f(x) \\
&\quad - \int_{\partial B(x_1,\rho)} \Gamma(x_1-s)[f(s+ix_2)-f(x_1+ix_2)]\, ds'_\alpha .
\end{aligned} \tag{4.22}
$$

(4.20) follows from the second line of (4.22) by letting $\rho \to 0$.

Let $V_\alpha(x)$ denote the integral in (4.20) and let

$$
V_{\rho\alpha}(x) = \int_{S_\rho(x)} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]\, d\xi = \int_{S_\rho(x)} \Gamma_{,\alpha}(x-\xi) f(\xi)\, d\xi ;
$$

we have seen that $V_{\rho\alpha}(x) \to V_\alpha(x)$ for $x \in B_{hR}$. Now let $\sigma = \rho_x/2$, $0 < \rho < \sigma$. Then, differentiating as above and using Cauchy's integral theorem, we obtain

$$
\begin{aligned}
V_{\rho\alpha,\beta}(x) &= \int_{B(x_1,\sigma)-B(x_1,\rho)} \Gamma_{,\alpha\beta}(x_1-s)[f(s+ix_2)-f(x_1+ix_2) \\
&\qquad - (s^\gamma - x_1{}^\gamma) f_{,\gamma}(x)]\, ds \\
&+ \int_{B_R-B(x_1,\sigma)} \Gamma_{,\alpha\beta}[(x_1-s)+i\lambda_\sigma x_2]\{f[s+i(1-\lambda_\sigma)x_2]-f(x)\} j_\sigma(s,x_1)\, ds \\
&- \int_{\partial B(x_1,\rho)} \Gamma_{,\alpha}(x_1-s)[f(s+ix_2)-f(x_1+ix_2)-(s^\gamma-x^\gamma) f_{,\gamma}(x)]\, ds'_\beta \\
&\qquad + C^\gamma{}_{\alpha\beta} f_{,\gamma}(x).
\end{aligned} \tag{4.23}
$$

Since f is analytic in B_{hR}, we may let $\rho \to 0$ to obtain

$$
\begin{aligned}
V_{\alpha,\beta}(x) &= C^\gamma{}_{\alpha\beta} f_{,\gamma}(x) + \int_{B(x_1,\sigma)} \Gamma_{,\alpha\beta}(x_1-s) \\
&\quad \times [f(s+ix_2)-f(x_1+ix_2)-(s^\gamma-x_1{}^\gamma) f_{,\gamma}(x)]\, ds \\
&\quad + \int_{S_\sigma(x)} \Gamma_{,\alpha\beta}(x-\xi)\,[f(\xi)-f(x)]\, d\xi,
\end{aligned} \tag{4.24}
$$

where the last integral in (4.24) is the middle one in (4.23).

Now, for $s \in B(x_1,\sigma)$, $d_\xi \geqq d_x/2$ where $\xi = s+ix_2$. Hence

$$
\| f(s+ix_2) - f(x) - (s^\gamma - x_1{}^\gamma) f_{,\gamma}(x) \| \leqq \tfrac{1}{2} K_8 2^{2-\mu} d_x{}^{\cdot} \, | s - x_1 |^2,
$$

using Lemma 4.6. Hence, since $| x_1 - s | \leqq \| x - \xi \| \leqq (\quad + h^2)^{\frac{1}{2}} | x_1 - s |$, we see that

$$|\nabla V_\alpha(x)| \leqq Z_1(h,\mu) h^*_\mu(f)\, d_x^{\mu-1}. \tag{4.25}$$

The desired Hölder continuity of V_α on $\bar{B}_{hR}$ follows from the fact that any two points y and z in B_{hR} can be joined by an arc $x = x(s)$, $0 \leqq s \leqq l$, such that

$$\int_0^l d_{x(s)}^{\mu-1}\, ds \leqq Z_2(h) \cdot \| y - z \|^\mu \cdot \mu^{-1}$$

The result (4.21) follows.

We now prove Theorem C.

THEOREM C. *Suppose u^k is of class $C^{t_k+\mu}$ for x real and on $\bar{B}_{hR}$ and satisfies* (1.9) *there with $f_j = 0$. Then u^k can be extended to be analytic on B_{hR} and of class $C^{t_k+\mu}$ on $\bar{B}_{hR}$ with*

$$h^*_\mu(\nabla^{t_k} u^k) \leqq K_3(h,\mu) h_\mu(\nabla^{t_k} u^k).$$

Proof. Let us define $U^k(x) = \int_{B_R} J(x-\xi) u^k(\xi)\, d\xi$, $x \in \bar{B}_R$. Then U^k is of class $C^{m+t_k+\mu}$ on $\bar{B}_R$ and satisfies $LU^k(x) = u^k(x)$ there. Let us define $F_j(x) = L_{jk} U^k(x)$, $x \in B_R$. Then if $m > 1$, we obtain

$$U^k{}_{,\alpha}(x) = \int_{B_R} J_{,\alpha}(x-\xi) u^k(\xi)\, d\xi = -\int_{B_R} u^k(\xi)\, (\partial/\partial\xi^\alpha) J(x-\xi)\, d\xi$$

$$= -\int_{\partial B_R} J(x-\xi) u^k(\xi)\, d\xi'_\alpha + \int_{B_R} J(x-\xi) u^k{}_{,\alpha}(\xi)\, d\xi;$$

one obtains the same result if $m = 1$ by approximating by U_ρ, etc., as in the proof of Lemma 3.7. By repeating this process, we see (since $L_{jk} u^k = 0$) that $F_j(x)$ is a sum of terms of the form

$$C \cdot \int_{\partial B_R} J_{,\alpha_1 \cdots \alpha_n}(x-\xi) u^k{}_{,\beta_1 \cdots \beta_p}(\xi)\, d\xi'_\gamma, \quad n + p + 1 = s_j + t_k$$

(there are no terms with $s_j + t_k = 0$). Now if the L^{qj} are defined as near (2.14), then $u^q(x) = L^{qj} F_j(x)$ and u^q is a sum of terms of the form

$$C \cdot \int_{\partial B_R} J_{,\alpha_1 \cdots \alpha_n}(x-\xi) u^k{}_{,\beta_1 \cdots \beta_p}(\xi)\, d\xi'_\gamma, \quad n + p = m + t_k - t_q - 1, \quad p \leqq t_k - 1.$$

If $n \geqq m-1$, the result follows from Lemma 4.5; if $n < m-1$, one can differentiate $m-1-n$ times and then apply Lemma 4.5.

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REFERENCES.

[1] S. Bernstein, "Sur la nature analytique des solutions des équations aux dérivées partièlles due second ordre," *Mathematische Annalen*, vol. 59 (1904), pp. 20-76.

[2] ———, "Demonstration du théorème de M. Hilbert sur la nature analytique des solutions des equations du type elliptique sans l'emploi des séries normales," *Mathematische Zeitschrift*, vol. 29 (1928), pp. 330-348.

[3] A. Douglas and L. Nirenberg, "Interior estimates for elliptic systems of partial differential equations," *Communications on Pure and Applied Mathematics*, vol. 8 (1955), pp. 503-538.

[4] A. Friedman, "Classes of solutions of linear non-hyperbolic differential equations," (to appear in the *Journal of Mathematics and Mechanics*).

[5] M. Gevrey, "Démonstration du théorème de Picard-Bernstein par la méthode des contours successifs; prolongement analytique," *Bulletin des Sciences Mathématiques*, (2) vol. 50 (1926), pp. 113-128.

[6] E. Hopf, "Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer differentialgleichungen zweiter ordnung," *Mathematische Zeitschrift*, vol. 34 (1932), pp. 194-233.

[7] E. E. Levi, "Sulle equazioni lineari totalemente ellittiche alle derivate parziali," *Palermo Rendiconti*, vol. 24 (1907), pp. 275-317.

[8] H. Lewy, "Neuer Beweis des analytischen Charakters der Lösungen elliptischer Differentialgleichungen," *Mathematische Annalen*, vol. 101 (1929), pp. 609-619.

[9] S. Mandelbrojt, *Séries de Fourier et classes quasi-analytiques de fonctions*, Collection Borel, Paris, 1935.

[10] C. B. Morrey, Jr. and L. Nirenberg, "On the analyticity of the solutions of linear elliptic systems of partial differential equations," *Communications on Pure and Applied Mathematics*, vol. 10 (1957), pp. 271-290.

[11] L. Nirenberg, "Remarks on strongly elliptic partial differential equations," *Communications on Pure and Applied Mathematics*, vol. 8 (1955), pp. 648-674.

[12] I. Petrowsky, "Sur l'analyticité des solutions des systèmes d'équations différentielles," *Rec. Math. N. S. Mat. Sbornik* 5(47), (1939), pp. 3-70.

[13] T. Radó, "Das Hilbertsche Theorem über den analytischen Charakter der Lösungen der partiellen Differentialgleichungen zweiter Ordnung," *Mathematische Zeitschrift*, vol. 25 (1926), pp. 514-589.

ON THE ANALYTICITY OF THE SOLUTIONS OF ANALYTIC NON-LINEAR ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS.*[1]

By CHARLES B. MORREY, JR.

PART II. Analyticity at the Boundary.

5. Introduction; strongly elliptic systems; outline of the proof. In this part, we shall show that a solution of a *strongly elliptic* (in the sense of Nirenberg, [5]) analytic system, which possesses Dirichlet data which are analytic along an analytic portion of the boundary of its domain of definition, can be extended analytically across that portion of the boundary.

The analyticity of the solutions of general analytic elliptic systems on the interior of their domains of definition has been proved in Part I [4] where a more extensive introduction and bibliography are given. We number the sections of Part II serially after those of Part I. Avner Friedman [1] has recently extended the results of Parts I and II to obtain corresponding results for quasi-analytic systems and the still more general systems in which the ϕ_j belong to the M_n classes of Mandelbrojt [3].

Since such a system remains analytic and strongly elliptic under an analytic transformation of the independent variables, we may choose any point on our portion of the boundary and transform the part of the domain and its boundary analytically onto that part of the hemisphere

$$(5.1) \qquad G_R: |x|^2 + y^2 < R^2, y > 0,\ x = (x^1, \cdots, x^\nu),\ \nu \geqq 1,\ R = R_0 > 0,$$

in such a way that the given point corresponds to the origin. Throughout this chapter, we shall let Σ_R be the part of the boundary ∂G_R of G_R for which $y > 0$ and σ_R denote that part for which $y = 0$. At times we shall let $X = (x, y)$, $X^\alpha = x^\alpha$, $\alpha \leqq \nu$, $X^{\nu+1} = y$.

The system (1.1) is said to be strongly elliptic at the vector function $u_0(X)$ if and only if the corresponding equations of variation form a strongly elliptic linear system as defined by Nirenberg in [5]:

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$$L_{jk}(X, D)u^k = f_j, \qquad j, k = 1, \cdots, N. \tag{5.2}$$

Here L_{jk} is an operator of order $\leqq s_j + t_k$, where we now assume, with Nirenberg, that $s_j = t_j \geqq 0$, $j = 1, \cdots, N$ and assume that there is an $m_2 > 0$ such that

$$\mathrm{Re}[L^0_{jk}(X, \Lambda)\xi^j\bar{\xi}^k] \geqq m_2 \sum_{j=1}^{N} |\Lambda|^{2s_j} |\xi_j|^2, \quad m_2 > 0 \tag{5.3}$$

for all real $\Lambda = (\lambda^1, \cdots, \lambda^{\nu+1})$ and all complex ξ; the coefficients in L may be complex; $L^0_{jk}(X, D)$ is the part of L_{jk} which is of order exactly $s_j + s_k$. The condition (5.3) guarantees that each operator L_{jj} is strongly elliptic of order $2s_j$. In case some $s_j = 0$, one may clearly solve the j-th equation of the non-linear system for u^j in terms of the others and the remaining system is easily seen to be strongly elliptic. Thus we shall assume that each $s_j > 0$ and shall define

$$s = \max s_j, \quad s_0 = \min s_j \qquad (0 < s_0 \leqq s). \tag{5.4}$$

Now, suppose P_0 is a given point of our boundary and u is a given solution of (1.1) in which u^k is of class $C^{s+s_k+\mu}$ $(0 < \mu < 1)$ near P_0 up to the boundary and such that

$$\nabla^p u^k(P) = g^{pk}(P), \qquad 0 \leqq p \leqq s_k - 1, \tag{5.5}$$

where the $g^{pk}(P)$ are analytic along the boundary near P_0. As in the first paragraph, we may assume $P_0 = 0$, σ_{R_0} to be the given portion of the boundary and $G_{R_0} \cup \sigma_{R_0}$ to be the given neighborhood of P_0. By subtracting off an analytic function of x only, we may assume that

$$D_y^p u^k(x, 0) = 0, \quad 0 \leqq p \leqq s_k - 1, \qquad k = 1, \cdots, N. \tag{5.6}$$

Definition. A vector u satisfying (5.6) will be said to possess 0 *Dirichlet data along* σ_{R_0}.

Next, we write

$$u^k(x, y) = Q^k(x, y) + v^k(x, y) \qquad (k = 1, \cdots, N), \tag{5.7}$$

where Q^k is that polynomial of degree $s + s_k$ which possesses 0 Dirichlet data along σ_{R_0} and such that

$$\nabla^p Q^k(0, 0) = \nabla^p u^k(0, 0), \qquad 0 \leqq p \leqq s + s_k. \tag{5.8}$$

Then, as in Section 1, we make the substitution (5.7) in the system (1.1) and rewrite the system in the form

$$L_{jk}v^k = M_{jk}v^k + \chi_j(x, y, \cdots, v^k, \nabla v^k, \cdots, \nabla^{s_j+s_k}v^k, \cdots), \tag{5.9}$$

where now L_{jk} is the operator of order exactly $s_j + s_k$ with coefficients evaluated at $(0,0)$, M_{jk} contains the lower order terms with constant coefficients, and χ_j is what remains. Since

$$\nabla^p v^k(0,0) = 0, \qquad 0 \leqq p \leqq s + s_k, \tag{5.10}$$

we may differentiate the j-th equation in (5.9) $s - s_j$ times with respect to the X^α and conclude that the Taylor expansion of χ_j begins with a homogeneous polynomial in X of degree $s - s_j + 1$, linear terms in the $\nabla^p v^k$ with coefficients homogeneous and linear in X, and quadratic terms in the $\nabla^p v^k$ with constant coefficients.

In addition to the notations of Part I, we shall use the following: As in (5.1), we often consider functions (or tensors) $u(x,y)$, $x = (x^1, \cdots, x^\nu)$, y being a single variable. Then $D^p u$ and $D_x^p u$ will denote any p-th partial derivative or any such involving only differentiation with respect to the x^α; such a derivative may also be denoted by $u_{,\alpha_1 \cdots \alpha_p}$; $\nabla_x^p u$ shall denote the tensor $u_{,\alpha_1 \cdots \alpha_p}$. For $0 < h < 1$, we let G_{hR} denote the set of all (x,y) such that y is real with $0 < y < R$ and $x = x_1 + ix_2$ is complex and $x \in B_{hR'}$ with $R' = \sqrt{(R^2 - y^2)}$.

The remainder of the proof is very similar to that in Section 1. In Section 7, we introduce a Green's function for the half-space E^+ where $y \geqq 0$, for the system

$$L_{jk} u^k = f_j \tag{5.11}$$

which enables us to prove the analogs, Theorems A′, B′, and C′, of the Theorems A, B, and C of Part I. We define the space E_R and $\| u \|$ as in Section 1 with $\bar{B}_R$ replaced by $\bar{G}_R$ and t_k replaced by $s + s_k$ and E_{R0} as the subset of those u in E_R which possess 0 Dirichlet data on σ_R. We also define the space F_R of vectors $f_j \in C^{s-s_j+\mu}$ with $\nabla^p f_j(0,0) = 0$ for $0 \leqq p \leqq s - s_j$ and $\| f \| = \sup_j h_\alpha(\nabla^{s-s_j} f_j)$. We next define the spaces E^*_{hR} and E^*_{hR0} as the subsets of E_R and E_{R0}, respectively, in which each $u^k(x,y)$ is analytic in x for $(x,y) \in G_{hR}$ and $u^k \in C^{s+s_k+\mu}$ in (x,y) on $\bar{G}_{hR}$ with $\| u \|^*_h = \sup_k h^*_\mu(\nabla^{s+s_k} u^k)$; the space F^*_{hR} is defined analogously.

The analogs of Theorems A, B, and C become:

THEOREM A′. *For each $R > 0$, there exists a bounded linear operator $P_{R0}(f)$ with bound $\leqq K_9(\mu)$ from F_R to E_{R0} such that, if $u = P_{R0}(f)$, then u is a solution of (5.11) on G_R.*

THEOREM B′. *There is a number $h_1 = h_1(\nu, N, s_j, m_2, M) > 0$ such that, if $f \in F_{hR}$ with $0 < h \leqq h_1$ and $u = P_{R0}(f)$, then u can be extended to $\in H^*_{hR0}$ on $\bar{G}_{hR}$ and $\| P_{R0}(f) \|^*_h \leqq K_{10}(\mu, h) \| f \|^*_h$.*

THEOREM C'. *If $H_R \in E_{R0}$ on $\bar{G}_R$ and H_R is a solution of the homogeneous equations* (5.11) *on G_R, then H_R can be extended to $\in E^*_{hR0}$ on any $\bar{G}_{hR}$ with $0 < h \leqq h_1$ with $\| H_R \|^*_h \leqq K_{11}(h, \mu) \| H_R \|$.*

The operator $T_R(V_R; H_R)$ can then be defined as in Section 1 and is seen to have the corresponding properties to (1.16) in the spaces E_{R0} and E^*_{hR0}. Thus, as in Section 1, we conclude that any solution v in E_{R0} on $\bar{G}_R$ of (5.9) $\in E^*_{hR0}$ on $\bar{G}_{hR}$ for any $h \leqq h_1$. But this implies that

$$| \nabla^n_x D_y^p u^k | \leqq A (LR^{-1})^n n!, \qquad n = 0, 1, 2, \cdots, 0 \leqq p \leqq s + s_k, \tag{5.12}$$

for all real (x, y) with $| x | \leqq R/2$ and $0 \leqq y \leqq R/2$. Since u is analytic for $y > 0$ by Part I, the proof of the Cauchy-Kowalewski theorem shows that corresponding bounds to (5.12) hold for *all* the derivatives including those with respect to y.

6. Certain systems of ordinary differential equations. In this section, we study certain solutions of systems of ordinary differential equations of the form

$$\sum_{k=1}^{N} \sum_{p=0}^{s_j + s_k} b_{jkp} \cdot (-i)^p D^p u^k(y) = f_j(y), \qquad j = 1, \cdots, N, \tag{6.1}$$

on the interval $[0, \infty)$. The coefficients are required to satisfy

$$\begin{gathered} \operatorname{Re} \sum_{j,k=1}^{N} \left[\sum_{p=0}^{s_j + s_k} b_{jkp} \mu^p \right] \xi^j \bar{\xi}^k \geqq m_2 \sum_{j=1}^{N} (1 + \mu^2)^{s_j} |\xi^j|^2, \qquad m_2 > 0, \\ \sum_{j,k=1}^{N} \sum_{p=0}^{s_j + s_k} | b_{jkp} |^2 \leqq M^2, \end{gathered} \tag{6.2}$$

for any real μ and any complex ξ; the coefficients may be complex.

We define H_0 as the set of all vectors u in which u^k is of class $C^{s_k - 1}$ with $D^{s_k - 1} u^k$ absolutely continuous and

$$D^p u^k(0) = 0, \qquad 0 \leqq p \leqq s_k - 1, \tag{6.3}$$

and for which the norm $\| u \|$ $(\geqq 0)$ defined by

$$\| u \|^2 = \int_0^\infty \sum_{j=1}^{N} \sum_{p=0}^{s_j} C_p^{s_j} | D^p u^j |^2 \, dy < \infty. \tag{6.4}$$

Clearly H_0 is a Hilbert space. For u and v having finite norm (not necessarily in H_0), we define

$$\begin{aligned} B(u, v) = \int_0^\infty \sum_{j,k=1}^{N} \{ \bar{v}^j \sum_{p=0}^{s_k} b_{jkp} (-i)^p D^p u^k \\ + \sum_{p=s_k+1}^{s_k + s_j} b_{jkp} (-i)^p (-1)^{p - s_k} D^{s_k} u^k D^{p - s_k} \bar{v}^j \} \, dy. \end{aligned} \tag{6.5}$$

We also define $B_a{}^b(u,v)$ in the same way with $[0,\infty)$ replaced by $[a,b]$.

LEMMA 6.1. *Any solution of the homogeneous equations* (6.1) *is the sum of exponential functions with possible polynomial coefficients. If* u *is a vector in which* u^k *is of class* C^{s_k-1} *on the interval* $[a,b]$ *with* $D^{s_k-1}u^k$ *absolutely continuous with* $D^{s_k}u^k$ *in* $\mathcal{L}_2$ *there and if* $B_a{}^b(u,v)=0$ *for every* v *of the same type with* $D^pv^k(a)=D^pv^k(b)=0$ *for* $0\leqq p\leqq s_k-1$, *then* u *is a solution of the homogeneous equations* (6.1).

Proof. The first statement is easily proved by restricting u to an arbitrary interval $[a,b]$ and then taking Fourier transforms. To prove the second, we note that a function v of the type required is uniquely determined by a vector w in $\mathcal{L}_2$ on $[a,b]$ in which

$$\int_a^b ((b-y)^p/p\,!)w^j(y)\,dy=0 \quad \text{for } p=0,\cdots,s_j-1,$$
$$v^j(y)=\int_a^y ((y-\eta)^{s_j-1}/(s_j-1)\,!)w^j(\eta)\,d\eta. \tag{6.6}$$

By inserting v^j as defined by (6.6) in $B_a{}^b(u,v)$ and interchanging order of integration and the letters y *and* η, we obtain

$$\int_a^b \sum_{j=1}^N A_j(y)\bar{w}_j(y)\,dy=0$$

for all w satisfying (6.6), $A_j(y)$ being defined in (6.7). It follows that there are constants d_{jp} such that

$$\begin{aligned} A_j(y)\equiv\sum_{k=1}^N \{&b_{jk,s_j+s_k}i^{s_j}(-i)^{s_k}D^{s_k}u^k(y)\\ &+\sum_{p=0}^{s_k} b_{jkp}(-i)^p\int_y^b ((\eta-y)^{s_j-1}/(s_j-1)\,!)D^pu^k(\eta)\,d\\ &+\sum_{p=s_k+1}^{s_j+s_k-1} b_{jkp}(-i)^p(-1)^{p-s_k}\\ &\times\int_y^b ((\eta-y)^{s_j+s_k-1-p}/(s_j+s_k-1-p)\,!)D^{s_k}u^k(\eta)\,d\eta\}\\ &=\sum_{p=0}^{s_j-1} d_{jp}(b-y)^p/p\,!. \end{aligned} \tag{6.7}$$

From (6.2), it follows that the determinant of the coefficients b_{jk,s_j+s_k} is not zero, so the $D^{s_k}u^k(y)$ can be expressed in terms of the other terms. Since these are all continuous we see first that the $D^{s_k}u^k$ are continuous and then that all the derivatives are continuous. Differentiating the j-th equation in (6.7) s_j times, $j=1,\cdots,N$, we see that u satisfies the homogeneous equations (6.1).

THEOREM 6.1. *Suppose that the $f_j(y)$ are of class C^{s-s_j} on $[0,\infty)$ and suppose $f_j(y)=0$ for $y\geqq Y$ (>0). Then there is a unique solution u^k of class C^{s+s_k} of (6.1) in which $u\in H_0$. Also u^k varies analytically with the coefficients b_{jkp} over the convex set defined by (6.2).*

Proof. Now, suppose u and $v\in H_0$. Introducing the Fourier transforms

$$U^k(z)=(2\pi)^{-\frac{1}{2}}\int_0^\infty e^{-iyz}u^k(y)\,dy, \qquad V^j(z)=(2\pi)^{-\frac{1}{2}}\int_0^\infty e^{-iyz}v^j(y)\,dy,$$

we see that

$$B(u,v)=\int_{-\infty}^{\infty}\sum_{j,k=1}^{N}\Big[\sum_{p=0}^{s_j+s_k}b_{jkp}z^p\Big]U^k\bar{V}^j\,dz. \tag{6.8}$$

From (6.8) and (6.2) we see that

$$\operatorname{Re}B(u,v)\geqq m_2\int^\infty\sum_{j=1}^{N}(1+z^2)^{s_j}\,|\,U^j\,|^2\,dz=m_2\,\|\,u\,\|^2. \tag{6.9}$$

Now, let Tu and w be defined in terms of u and f by

$$(Tu,v)=B(u,v),\ (w,v)=\int_0^\infty f_j\bar{v}^j\,dy,\ v\in H_0. \tag{6.10}$$

From (6.2), it follows that T is a bounded linear operator. From a lemma of Lax and Milgram ([2], p. 169), it follows that T has an inverse and that $\|\,T^{-1}\,\|\leqq m_2^{-1}$. Now the equations $Tu=v$ and

$$B(u,v)=\int_0^\infty f_j\bar{v}^j\,dy \text{ for all } v\in H_0 \tag{6.11}$$

are equivalent. Accordingly there is a unique solution u in H_0 of (6.11). Since T is analytic (linear, in fact) in the coefficients b_{jkp}, it follows that u is for each fixed f.

Finally, if we write the equations (6.1) in the form $l_{jk}u^k=f_j$, let $l(\mu)$ be the determinant of the $l_{jk}(\mu)$, let F_j be a solution of $l(D)F_j=f_j$, and set $u_1{}^k=l^{kl}F_l$, where $l^{jk}(\mu)$ is the cofactor of $l_{kj}(\mu)$; then u_1 is a solution of (6.1) with the stated differentiability properties. Obviously, this solution satisfies (6.11) with the interval $[0,\infty)$ replaced by $[0,Y]$ for any Y. Clearly the difference $u-u_1$ satisfies $B_0{}^Y(u-u_1,v)=0$ for all v as in Lemma 6.1 so that $u-u_1$ is a solution of the homogeneous equations (6.1) Thus u has the differentiability properties as stated.

THEOREM 6.2. *There exist unique vector solutions $u_m{}^q$ $(=u_m{}^{kq})$ of the homogeneous equations (6.1), each having finite norm as defined in (6.4) and satisfying the initial conditions*

$$D^p u_m{}^{kq}(0) = \delta_m{}^k \delta_q{}^p, \quad k, m = 1, \cdots, N, \quad 0 \leqq p \leqq s_k - 1, \quad 0 \leqq q \leqq s_m - 1. \tag{6.12}$$

These solutions depend analytically on the coefficients b_{jkp} on the convex set of b_{jkp} satisfying (6.2).

Proof. For, let $u_{0m}{}^q$ be any vector class C^∞ vanishing for $y \geqq 1$ and satisfying the given initial conditions; we assume that $u_{0m}{}^q$ is independent of the b_{jkp}. From Theorem 6.1, it follows that there exists a unique solution $u_{1m}{}^q$ in H_0 of equations (6.1) with $f_{mj}{}^q = L_{jk} u_{0m}{}^{kq}$. The vector $u_m{}^q = u_{0m}{}^q - u_{1m}{}^q$ satisfies the given conditions. If $u_{2m}{}^q$ were another such, then $u_{2m}{}^q - u_m{}^q$ would satisfy the homogeneous equations (6.1) and would belong to H_0 and would therefore vanish by Theorem 6.1.

LEMMA 6.2. *There are rectangles R^+, $R_1{}^+$, R^-, and $R_1{}^-$ with $R^+ \cup \partial R^+$ in the upper half-plane, with $R^- \cup \partial R^-$ symmetrically located in the lower half-plane, with $R_1{}^\pm \cup \partial R_1{}^\pm$ interior to $R^\pm$ and symmetrically placed, which depend only on ν, N, s_j, m_2, and M, such that for any set of coefficients satisfying* (6.2), *all the roots of $L(z) = 0$* (see below) *with positive imaginary parts lie in $R_1{}^+$ and all those with negative imaginary parts lie in $R_1{}^-$, there being $s_1 + \cdots + s_N$ roots of each type.*

Proof. This is obvious since the set of all complex coefficients satisfying (6.2) is compact and convex, the roots of $L(z) = 0$ vary continuously with the coefficients, no root is real, and the set contains the particular array of coefficients

$$b_{jj,2q} = C_q{}^{s_j}, \qquad b_{jkp} = 0 \text{ otherwise.}$$

Now, let us define the functions

$$\hat{u}_m{}^{kq}(z) = (2\pi)^{-1} \int_0^\infty u_m{}^{kq}(u) e^{-iuz}\, du, \quad \Im(z) < \epsilon, \tag{6.13}$$

$\hat{u}_m{}^{kq}(z)$ being defined wherever possible by analytic extension, and let

$$L_{jk}(z) = \sum_{p=0}^{s_j + s_k} b_{jkp} z^p, \tag{6.14}$$

L^{kl} be the cofactor of L_{lk} and

$$L(z) = \det \| L_{jk}(z) \|, \qquad M^{kl}(z) = L^{-1}(z) L^{kl}(z). \tag{6.15}$$

Then we have the following results:

THEOREM 6.3. *The $\hat{u}_m{}^{kq}$ are rational functions of z which are analytic*

for z outside R_1^+ and whose coefficients vary analytically with the coefficients b_{jkp} over the set (6.2). We have

$$u_m^{kq}(u) = \int_{\partial R^+} e^{iuz}\hat{u}_m^{kq}(z)\,dz, \quad u \geqq 0, \tag{6.16}$$

$$L_{jk}(z)\hat{u}_m^{kq}(z) = P_{jm}^{q}(z), \tag{6.17}$$

$$\int_{\partial R^+} (iz)^p \hat{u}_m^{kq}(z)\,dz = \delta^{pq}\delta_m^{\ k} \text{ if } 1 \leqq k, m \leqq N,\ 0 \leqq p < s_k, \tag{6.18}$$
$$0 \leqq q < s_m,$$

$$(iz)^p M^{kl}(z) - \sum_{m=1}^{N}\sum_{q=0}^{s_m-1} \hat{u}_m^{kq}(z) \int_{\partial R^+} (i\zeta)^{p+q} M^{ml}(\zeta)\,d\zeta = P^{pkl}(z), \tag{6.19}$$

where the P_{jm}^{q} are polynomials (or $\equiv 0$) and the P^{pkl} are rational functions which are analytic outside of R_1^-.

Proof. The first statement follows from the form of the u_m^{kq} and their analytic dependence on the coefficients and (6.13). (6.16) also follows by writing out what u_m^{kq} is explicitly. To prove (6.17), we use the facts that the u_m^{kq} are solutions of the homogeneous equations (6.1) and the $\hat{u}_m^{kq}$ are rational functions. (6.18) follows from (6.16) and (6.12). To prove (6.19), let P^{pkl} denote the left side of (6.19) and define

$$v^{kl}(u) = \int_{\partial R^+} e^{iuz} M^{kl}(z)\,dz, \quad U^{pkl}(u) = \int_{\partial R^+} e^{iuz} P^{pkl}(z)\,dz. \tag{6.20}$$

We see that the vector v^l and all of its derivatives are solutions of finite norm of the homogeneous equations (6.1) and then that

$$U^{pkl}(u) = D^p v^{kl}(u) - \sum_{m=1}^{N}\sum_{q=0}^{s_m-1} u_m^{kq}(u) D^{p+q} v^{ml}(0), \tag{6.21}$$

so that the vectors U^{pl} are also solutions of finite norm of the homogeneous equations (6.1). Also

$$D^r U^{pkl}(0) = D^{p+r} v^{kl}(0) - \sum_{m=1}^{N}\sum_{q=0}^{s_m-1} D^r u_m^{kq}(0) D^{p+q} v^{ml}(0) = 0, \quad 0 \leqq r < s_k,$$

by virtue of (6.12). Hence the $U^{pkl}(u) \equiv 0$ from which the result follows.

7. The Green's matrix; proof of Theorems A′, B′, and C′. In this section, we define the Green's matrix alluded to in Section 5 and prove Theorems A′, B′, and C′. We may write the operator

$$L_{jk} = \sum_{p=0}^{s_j+s_k} \sum_{\alpha} a_{jkp}^{\alpha_1 \cdots \alpha_p} D_y^{s_j+s_k-p} D_x^{\alpha_1} \cdots D_x^{\alpha_p} \tag{7.1}$$

and we define the operator

$$L^*_{jk} = (-1)^{s_j+s_k} L_{jk}. \tag{7.2}$$

If u^k and v^j are sufficiently smooth on a sufficiently smooth domain D, we see that

$$\int_D \sum_{j,k=1}^{N} [v^j L_{jk} u^k - u^k L^*_{jk} v^j] dxdy = \int \sum_{\alpha=1}^{\nu+1} B^\alpha(u,v) n_\alpha \, dS, \tag{7.3}$$

where each $B^\alpha(u,v)$ is a sum of terms of the form

$$cD^p u^k D^q v^j, \text{ where } p+q = s_j + s_k - 1; \tag{7.4}$$

the B^α are not unique but we select particular ones.

We define the functions ${u_m}^{kq}(z,\sigma)$, $L_{jk}(z,\sigma)$, $L^{kl}(z,\sigma)$, $M^{kl}(z,\sigma)$ and $L(z,\sigma)$ by (6.3) through (6.15) with the b_{jkp} replaced by

$$b_{jkp}(\sigma) = \sum_{\alpha} a_{jkn}{}^{\alpha_1 \cdots \alpha_n} \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}, \; n = s_j + s_k - p, \; \sigma = (\sigma_1, \cdots, \sigma_\nu). \tag{7.5}$$

For any σ with $|\sigma| = 1$, we see that

$$\begin{aligned} \operatorname{Re} \sum_{j,k=1}^{N} \left[\sum_{p=0}^{s_j+s_k} b_{jkp}(\sigma) \mu^p \right] \xi^j \bar{\xi}^k &= \operatorname{Re} \sum_{j,k=1}^{N} L_{jk}(\sigma,\mu) \xi^j \bar{\xi}^k \\ &\geqq m_2 \sum_{j=1}^{N} (1+\mu^2)^{s_j} |\xi_j|^2, \; m_2 > 0 \end{aligned} \tag{7.6}$$

by virtue of the strong ellipticity condition (5.3). Thus (6.2) holds for all such σ and we see that all the ${\hat{u}_m}^{kq}(z,\sigma)$ are also analytic in σ near σ real with $|\sigma| = 1$.

Next, we define the functions $J_h(w)$ by

$$\begin{aligned} J_h(w) &= \begin{cases} h!(-iw)^{-h-1}, & h \geqq 0 \\ ((iw)^{-h-1}/(-h-1)!)[-\log(-iw) + C_{-h-1}], & h < 0 \end{cases} \\ C_0 &= 0, \; C_t = 1 + 1/2 + \cdots + 1/t \text{ if } t \geqq 1, \end{aligned} \tag{7.7}$$

where for the log, we cut the w plane along the negative imaginary axis and take the principal log. We note that

$$D^p J_h(w) = i^p J_{h+p}(w). \tag{7.8}$$

Now, let us define

$$\begin{aligned} \Gamma^{kl}(x,y) &= C_1{}^{kl} \int_\Sigma \{ \pm \int_{\partial R^\pm} M^{kl}(z,\sigma) J_{\nu-s_k-s_l}(\sigma \cdot x + yz) \, dz \} d\Sigma_\sigma \\ C_1{}^{kl} &= (2\pi)^{-\nu-1} i^{-s_k-s_l}, \; \Sigma : |\sigma| = 1, \; y \neq 0, \end{aligned} \tag{7.9}$$

the sign being that of y (defining $\Gamma^{kl}(x,0)$ by continuity),

$$H^{kl}(x,y,\eta) = C_1{}^{kl}\int_\Sigma d\Sigma \int_{\partial R^+} dz \int_{\partial R^-} N^{kl}(z,\zeta,\sigma) \times J_{\nu-s_k-s_l}(\sigma\cdot x + yz - \eta\zeta)\,d\zeta, \tag{7.10}$$

$$N^{kl}(z,\zeta,\sigma) = \sum_{m=1}^{N}\sum_{q=0}^{s_m-1} \hat{u}_m{}^{kq}(z,\sigma)\,(i\zeta)^q M^{ml}(\zeta,\sigma),$$

$$G^{kl}(x,y,\eta) = \Gamma^{kl}(x,y-\eta) + H^{kl}(x,y,\eta), \tag{7.11}$$

$$J^*(x,y) = (-1)^{S+P}(2\pi)^{-\nu-1}\int_\Sigma \{\pm\int_{\partial R^\pm}(1+z^2)^{-P}L^{-1}(z,\sigma) \times J_{\nu-2S-2P}(\sigma\cdot x + yz)\,dz\}d\Sigma, \tag{7.12}$$

$$M(x,y) = LJ^*(x,y), \tag{7.13}$$

$$\Gamma^{*kl}(x,y) = L^{kl}J^*(x,y), \tag{7.14}$$

where, in (7.12), P is taken so large that $\nu + s_k - s_l - 2P < -1$ for all k,l.

Remark. The case $\nu = 1$ is special in the sense that Σ consists of the two points $\sigma = \pm 1$ and we interpret $\int_\Sigma f(\sigma,\cdots)d\Sigma$ in this case as simply $f(1,\cdots)+f(-1,\cdots)$ in (7.9), (7.10), and (7.12). In what follows, we shall treat only the case $\nu > 1$ leaving the case $\nu = 1$ to the reader.

Definition. A function $f(x)$ is said to be *essentially homogeneous of degree* p if and only if $p < 0$ implies that f is positively homogeneous of degree p and $p \geqq 0$ implies that

$$f(X) = f_0(X)\log|X| + f_1(X), \tag{7.15}$$

where f_1 is positively homogeneous of degree p and f_0 is a homogeneous polynomial of degree p (or $\equiv 0$).

THEOREM 7.1. *There is an* $h_2(\nu, N, s_j, m_2, M) > 0$ *such that* $J^*(X)$ *is analytic for all complex* $X = X_1 + iX_2 \neq 0$ *for which* $|X_2| \leqq h_2|X_1|$. *For such* X, J^* *is essentially homogeneous of degree* $2S + 2P - \nu - 1$. *Furthermore*

$$\Delta^{P-1}M(X) = \Gamma_0(X),\ \Delta^P M(X) = 0, \tag{7.16}$$

$$\Delta^P\Gamma^{*kl}(X) = \Gamma^{kl}(X), \qquad X \neq 0 \tag{7.17}$$

$$L_{jk}\Gamma^{*kl}(X) = L_{kj}\Gamma^{*lk}(X) = \delta_j{}^l M(X), \tag{7.18}$$

Proof. The essential homogeneity of J^* is obvious. If $y \neq 0$, we may replace $\pm\int_{\partial R^\pm}$ by $\int_{-\infty}^{\infty}$ in (7.12). By introducing new variables τ on the

unit sphere S in $(\nu+1)$ space by

$$\tau^\alpha = \sigma^\alpha(1+z^2)^{-\frac{1}{2}}, \qquad \tau^{\nu+1} = z(1+z^2)^{-\frac{1}{2}}$$

and noting that $dzd\Sigma = (1+z^2)^{(\nu+1)/2}\,dS$, we see that we may write J^* in the form

$$\begin{aligned} J^*(X) = &(2\pi i)^{-\nu-1}\int_S L^{-1}(\tau)\cdot((\tau\cdot X)^{2S+2P-\nu-1}/(2S+2P-\nu-1)!) \\ &\times[C_{2S+2P-\nu-1}+\log(1-(\tau^{\nu+1})^2)^{\frac{1}{2}}-\log(-i\tau\cdot X)]\,dS_\tau. \end{aligned} \tag{7.19}$$

The first two terms in the bracket in (7.19) yield homogeneous polynomials in X of the desired degree. The analyticity of J^* follows as in Section 3.

Using (7.12) and (7.14), we see that

$$\begin{aligned} \Gamma^{*kl}(x,y) = (-1)^P C_1{}^{kl}\int_\Sigma\{\pm\int_{\partial R^\pm}(1+z^2)^{-P}M^{kl}(z,\sigma) \\ \times J_{\nu-s_k-s_l-2P}(\sigma\cdot x+yz)\,dz\}d\Sigma, \end{aligned} \tag{7.20}$$

from which (7.17) follows easily. Next,

$$\begin{aligned} \sum_k L_{jk}\Gamma^{*kl}(x,y) &= Z_1\int_\Sigma\{\pm\int_{\partial R^\pm}(1+z^2)^{-P}\delta_j{}^l J_{\nu+s_j-s_l-2P}(\sigma\cdot x+yz)dz\}d\Sigma \\ &= \delta_j{}^l M(x,y),\ Z_1 = (-1)^P i^{s_j-s_l}(2\pi)^{-\nu-1}. \end{aligned}$$

Likewise

$$\sum_k L_{kj}\Gamma^{*lk}(x,y) = \delta_j{}^l M(x,y)$$

$$\begin{aligned} M(x,y) &= (-1)^P(2\pi)^{-\nu-1}\int_\Sigma\{\pm\int_{\partial R^\pm}(1+z^2)^{-P}J_{\nu-2P}(\sigma\cdot x+yz)dz\}d\Sigma \\ &= (2\pi i)^{-\nu-1}\int_S((\tau\cdot X)^{2P-\nu-1}/(2P-\nu-1)!) \\ &\quad\times[C_{2P-\nu-1}+\log\sqrt{[1-(\tau^{\nu+1})^2]}-\log(-i\tau\cdot X)]\,dS_\tau \end{aligned} \tag{7.21}$$

To prove (7.16), we first consider the case $\nu = 2k$. Then

$$\begin{aligned} \Delta^{P-k-1}M(X) = (2\pi i)^{-\nu-1}&\int_S(\tau\cdot X) \\ &\times[1+\log\sqrt{[1-(\tau^{\nu+1})^2]}-\log(-i\tau\cdot X)]\,dS_\tau. \end{aligned}$$

In order to evaluate this, we set $\tau\cdot X = |X|\cos\phi$ and the integral becomes

$$\begin{aligned} -(2\pi i)^{-\nu-1}\Gamma_\nu|X|&\int_0^\pi\sin^{2k-1}\phi\cos\phi\log(-i\cos\phi)\,d\phi, \\ &\Gamma_\nu = \nu\pi^{\nu/2}/\Gamma(1+\nu/2). \end{aligned} \tag{7.22}$$

In (7.22), we may replace the segment from 0 to π by any path with the

same endpoints for which $\mathfrak{I}\cos\phi \geqq 0$ and >0 when $R\cos\phi = 0$. Then, integrating by parts, etc., (7.22) becomes

$$-(2\pi i)^{-2k-1}(2k)\pi^k/k!\cdot(-\pi i/2k)\,|X| = (-1)^k(\pi^{-k}\cdot 2^{-2k-1}/k!)\,|X|.$$

Thus

$$\Delta^{P-1}M(X) = -2^k\cdot k!\cdot 1\cdot 3\cdot\cdots(2k-3)\,r^{1-2k}/\pi^k 2^{2k+1}k! = \Gamma_0(X)$$

$$(\Gamma_0(X) = -(\Gamma(k+3/2)/(2k-1)(2k+1)\pi^{k+1/2})\,|X|^{1-2k}, \text{ see } (2.9))$$

in this case, The case where $\nu = 2k+1$ is treated similarly.

LEMMA 7.1. *The function*

$$F_{vrh}(z,\zeta,\sigma,A) = \int_{B_A}\xi^{\beta_1}\cdot\cdots\cdot\xi^{\beta_r}\,d\xi \times\int_0^\infty \eta^h J_{\nu+2+v}(-\sigma\cdot\xi+z-\eta\zeta)\,d\eta,\ r+h\leqq v+1, \tag{7.23}$$

is uniformly bounded for all z on ∂R^+, ζ on ∂R^-, $|\sigma|=1$, and real $A\geqq 0$ and tends to zero as $A\to+\infty$ for each set of (z,ζ,σ) as described.

Proof. Since $\mathfrak{I}(z-\sigma\cdot\xi)>0$ and $\mathfrak{I}(\zeta)<0$, we see that

$$\int_0^\infty \eta^h J_{\nu+2+v}(-\sigma\cdot\xi+z-\eta\zeta)\,d\eta = Z_1(-\zeta)^{-h-1}(z-\sigma\cdot\xi)^{h-\nu-2-v}.$$

To evaluate

$$\int_{B_A}\xi^{\beta_1}\cdot\cdots\cdot\xi^{\beta_r}\,(z-\sigma\cdot\xi)^{h-\nu-2-v}\,d\xi \tag{7.24}$$

for a given σ with $|\sigma|=1$, first make a rotation of axes from ξ to $'\xi$ so that $\sigma\cdot\xi = {}'\xi^1$ and then set $|\xi| = |{}'\xi| = \rho$, ${}'\xi^1 = \rho\cos\phi$ and the integral in (7.24) becomes a sum of terms of the form

$$c(\sigma)\int_0^\pi \sin^{\nu-2+s}\phi\cos^{r-s}\phi\,d\phi\int_0^A \rho^{r+\nu-1} \times(-\rho\cos\phi+z)^{h-\nu-2-v}\,d\rho. \tag{7.25}$$

In each of the terms in (7.25), we may replace the segment from 0 to π by a path γ such that $-\cos\phi$ describes the upper half circle from -1 to 1. Then (7.25) becomes

$$c_1(\sigma)z^{r+h-2-\nu}\int_\gamma \tan^{\nu-2+s}\phi\sec^2\phi \times\left[\int_0^{-A\cos\phi/z}\tau^{r+\nu-1}(\tau+1)^{h-\nu-2-v}\,d\tau\right]d\phi. \tag{7.26}$$

Since $r+h\leqq v+1$, the result follows from the form (7.26).

THEOREM 7.2. *There is an* $h_1(\nu, N, s_j, m_2, M)$ *with* $0 < h_1 \leqq h_2$ *such that* $\Gamma^{kl}(x,y)$ *is analytic for all complex* $X = X_1 + iX_2 \neq 0$ *with* $|X_2| \leqq h_1 |X_1|$ *and* $H^{kl}(x,y,\eta)$ *is analytic for all complex*

$$(x_1 + ix_2, y_1 + iy_2, \eta_1 + i\eta_2) \neq (0,0,0)$$

such that $y_1 \geqq 0$, $\eta_1 \geqq 0$, *and*

$$(|x_2|^2 + y_2^2 + \eta_2^2) \leqq h_1^2(|x_1|^2 + y_1^2 + \eta_1^2).$$

Γ^{kl} *and* H^{kl} *are essentially homogeneous of degree* $s_k + s_l - \nu - 1$ *in their arguments on the domain above. We also have the formulas*

$$L_{jk}(X)\Gamma^{kl}(X) = L_{jk}(X)H^{kl}(X,\Xi) = 0; \tag{7.27}$$

$$L^*_{jk}(\Xi)\Gamma^{lj}(X-\Xi) = L^*_{jk}(\Xi)H^{lj}(X,\Xi) = 0; \tag{7.28}$$

$$D_y{}^p G^{kl}(x,0,\eta) = D_\eta{}^r G^{kl}(x,y,0) = 0,\ \eta > 0, y > 0,\ 0 \leqq p < s_k,\ 0 \leqq r < s_l. \tag{7.29}$$

$$\int_{E^+} D^2 D_y{}^v \omega(-\xi, y, \eta)\eta^h \xi^{\beta_1} \cdots {}^{\beta_r} d\xi d\eta = 0 \text{ for } h + r \leqq v + 1, \tag{7.30}$$

$h, r \geqq 0$, where $\omega = D^{s_k + s_l - 1}(H_y{}^{kl} + H_\eta{}^{kl})$.

Finally, if $f_j \in C^{s-s_j+\mu}$ *on* $\bar{G}_a$ *and* u *is a solution of* (5.11) *in which* $u^k \in C^{s+s_k+\mu}$ *on* $\bar{G}_a$, *then*

$$u^l(X) = \int_{G_a} G^{lj}(X,\Xi) f_j(\Xi)\, d\Xi - \int_{\partial G_a} B^\alpha[G^l(X,\Xi), u(\Xi)]\, n_\alpha\, dS. \tag{7.31}$$

Proof. The analyticity of Γ^{kl} follows from Theorem 7.1. The analyticity of H^{kl} for real $(x,y,\eta) \neq (0,0,0)$ with $y \geqq 0$, $\eta \geqq 0$, $y + \eta > 0$ is obvious from (7.10). To remove the restriction y or $\eta > 0$ with $x \neq 0$, let $x_0 \neq 0$ and make a rotation $\sigma = \sigma(\tau, \xi)$, $\xi = x/|x|$, from σ to τ which is analytic in x for x near x_0 so that $\sigma \cdot x = |x|\tau^1$, then let

$$\tau^1 = \cos\phi,\ \tau^{1+\alpha} = \omega^\alpha \sin\phi,\ |\omega| = 1,$$

$$N_1{}^{kl}(z,\zeta,\phi,\xi) = \sin^{\nu-2}\phi \int_{|\omega|=1} N^{kl}[z,\zeta,\sigma(\cos\phi, \omega\sin\phi, \xi)]\, dS_\omega;$$

the expression for H^{kl} then becomes

$$C_1{}^{kl} \int_0^\pi d\phi \int_{\partial R^+} dz \int_{\partial R^-} N_1{}^{kl}(z,\zeta,\phi,\xi) J_{\nu-s_k-s_l}(|x|\cos\phi + yz - \eta\zeta)\, d\zeta. \tag{7.32}$$

The segment $[0,\pi]$ in (7.32) may be replaced by a nearby path γ on which $\Re \cos\phi > 0$ except at the endpoints, N_1 being analytic in its arguments. The essential homogeneity is obvious.

The first of the formulas in (7.27) and (7.28) follow from Theorem 7.1. Since $i^{s_j+s_k}C_1{}^{kl} = i^{s_j-s_l}(2\pi)^{-\nu-1}$ and

$$L_{jk}(z,\sigma)N^{kl}(z,\zeta,\sigma) = \sum_{m,q} (i\zeta)^q M^{ml}(\zeta,\sigma)P^q{}_{jm}(z,\sigma)$$

we see that the second formula of (7.27) holds. Now

$$L^*{}_{jk}(\xi,\eta)H^{lj}(x-\xi,y,\eta) = i^{s_k-s_l}(2\pi)^{-\nu-1}\int_\Sigma d\Sigma \int_{\partial R^+} dz$$
$$\times \int_{\partial R^-} L_{jk}(\zeta,\sigma)N^{lj}(z,\zeta,\sigma)J_{\nu+s_k-s_l}(\sigma\cdot x + yz - \eta\zeta)\,d\zeta = 0,$$

since $L_{jk}(\zeta,\sigma)N^{lj}(z,\zeta,\sigma) = \sum_{q=0}^{s_k-1} \hat{u}_k{}^{lq}(z,\sigma)\,(i\zeta)^q$.

Now, the first of (7.29) follows easily since

$$\int_{\partial R^+} (iz)^p N^{kl}(z,\zeta,\sigma)\,dz$$
$$= \sum_{m,q} (i\zeta)^q M^{ml}(\zeta,\sigma) \int_{\partial R^+} (iz)^p \hat{u}_m{}^{kq}(z,\sigma)\,dz = (i\zeta)^p M^{kl}(\zeta,\sigma)$$

using (7.10) and (6.18). The second result in (7.29) follows from (7.10) and (6.19) with $p = r$ since

$$\int_{\partial R^-} (i\zeta)^r N^{kl}(z,\zeta,\sigma)\,d\zeta = -\int_{\partial R^+} (i\zeta)^r N^{kl}(z,\zeta,\sigma)\,d\zeta$$

on account of the restrictions on q and r and the degrees of the rational functions involved.

To prove (7.30), we note that

$$D^2 D_y{}^v \omega(-\xi,y,\eta)$$
$$= \int_\Sigma d\Sigma \int_{\partial R^+} dz \int_{\partial R^-} \phi(z,\zeta,\sigma)J_{\nu+2+v}(-\sigma\cdot\xi + yz - \eta\zeta)\,d\zeta,$$

where ϕ is a polynomial in (z,ζ,σ) times N^{kl}. On account of the homogeneity, we may assume $y = 1$; and the integral in (7.30) is absolutely convergent so that it may be written in the form

$$\lim_{A\to\infty} \int_\Sigma d\Sigma \int_{\partial R^+} dz \int_{\partial R^-} \phi(z,\zeta,\sigma)F_{vrh}(z,\sigma,A)\,d\zeta = 0$$

by Lemma 7.1.

To prove (7.31), we first notice that all the derivatives occurring in $L_{jk}(X)\Gamma^{*lj}(X-\Xi)$ are continuous at $\Xi = X$ on account of our choice of P in (7.12) and we may apply (7.3) with $D = G_a$ and $v^j = \Gamma^{*lj}$ to obtain

$$(7.33)\qquad \int_{G_a} M(X-\Xi)u^l(\Xi)\,d\Xi = \int_{G_a} \Gamma^{*lj}(X-\Xi)f_j(\Xi)\,d\Xi - \int_{\partial G_a} B^\alpha[\Gamma^{*l}(X-\Xi), u(\Xi)]n_\alpha\,dS_\Xi$$

since

$$L^*_{jk}(\Xi)\Gamma^{*lj}(X-\Xi) = L_{jk}(X)\Gamma^{*lj}(X-\Xi) = \delta_k{}^l M(X-\Xi).$$

By operating on both sides of (7.33) with $\Delta^P(X)$, we obtain

$$(7.34)\qquad u^l(X) = \int_{G_a} \Gamma^{lj}(X-\Xi)f_j(\Xi)\,d\Xi - \int_{\partial G_a} B^\alpha[\Gamma^l(X-\Xi), u(\Xi)]n_\alpha\,dS_\Xi.$$

Applying (7.3) with $D = G_a$ and $v^j = H^{lj}$, we obtain

$$(7.35)\qquad 0 = \int_{G_a} H^{lj}(X-\Xi)f_j(\Xi)\,d\Xi - \int_{\partial G_a} B^\alpha[H^l(X,\Xi), u(\Xi)]n_\alpha\,dS_\Xi.$$

The result follows by adding (7.34) and (7.35).

THEOREM 7.3. *Suppose $f_j(x,y) \in C^{s-s_j+\mu}$ on E^+ and vanishes outside G_a and suppose that*

$$(7.36)\qquad u^k(x,y) = \int_{E^+} G^{kl}(x-\xi, y, \eta)f_l(\xi,\eta)\,d\xi d\eta \text{ for } y > 0.$$

Then $u^k \in C^{s+s_k+\mu}$ on E^+, u is a solution of (5.11) for $y > 0$, u has 0 Dirichlet data on $y = 0$ and

$$\sup_k h_\mu(\nabla^{s+s_k}u^k) \leqq K_{12}(\mu) \sup_l h_\mu(\nabla^{s-s_l}f_l).$$

If, also, $f_j(x,y)$ is analytic in x for $(x,y) \in G_{hR}$ and $\in C^{s-s_j+\mu}$ on $\bar{G}_{hR}$ for some R with $0 < R \leqq a$ and h with $0 < h \leqq h_1$, then u^k can be extended to be analytic in x on G_{hR} and of class $C^{s-s_k+\mu}$ on G_{hR} and

$$\sup_k h^*_\mu(\nabla^{s+s_k}u^k) \leqq K_{13}(h,\mu) \sup_l h^*_\mu(\nabla^{s-s_l}f_l) \text{ on } \bar{G}_{hR}.$$

Proof. From the essential homogeneity of Γ^{kl} and H^{kl}, it follows that we can differentiate (7.36) p times $(0 \leqq p \leqq s_k - 1)$ with respect to y and set $y = 0$ under the integral sign; that u^k has 0 Dirichlet data then follows from (7.29).

Next choose (x_0, y_0) and ρ with $0 < \rho < y_0$ and let $u^k(x,y) = v^k(x,y) + w^k(x,y)$, where

$$w^k(x,y) = \int_{B(x_0,y_0,\rho)} \Gamma^{kl}(x-\xi, y-\eta)f_l(\xi,\eta)\,d\xi d\eta.$$

It is easy to see that v is analytic inside $B(x_0, y_0, \rho)$ and is a solution of the homogeneous equations (5.11) there and that

$$(7.37) \qquad w^k(X) = \sum_l \Delta^P L^{kl} F_l(X), F_l(X) = \int_{B(X_0,\rho)} J^*(X-\Xi) f_l(\Xi)\, d\Xi.$$

Using the methods of Part I, one easily establishes that $F_l \in C^{2P+2S+s-s_l+\mu}$ $(S = s_1 + \cdots + s_N)$ so that $w^k \in C^{s+s_k+\mu}$. From (7.37), the essential homogeneity, and Theorem 7.1, we see that

$$L_{jk} w^k(X) = \sum_l \Delta^P [L_{jk} L^{kl} F_l(X)] = \Delta^P L F_j(X)$$
$$= \Delta^P \int_{B(X_0,\rho)} M(X-\Xi) f_j(\Xi)\, d\Xi = f_j(X).$$

Thus $u^k \in C^{s+s_k+\mu}$ and u is asolution of (5.11) for $y > 0$.

To show that $u^k \in C^{s+s_k+\mu}$ for $y \geqq 0$, we differentiate each term (l fixed) of (7.36) $s + s_k - 1$ times. We note that as long as ϕ is differentiable, we may write

$$D_{x^\alpha} \int_{E^+} G^{kl}(x-\xi, y, \eta)\phi(\xi,\eta)\, d\xi d\eta = -\int_{E^+} D_{\xi^\alpha} G^{kl} \phi\, d\xi d\eta$$
$$= \int_{E^+} G^{kl}(x-\xi, y, \eta) D_{\xi^\alpha}\phi(\xi,\eta)\, d\xi d\eta$$

$$(7.38) \qquad D_y \int_{E^+} G^{kl}\phi\, d\xi d\eta = \int_{E^+} [\Omega^{kl}\phi + G^{kl} D_\eta \phi(\xi,\eta)]\, d\xi d\eta,$$
$$\Omega^{kl} = D_y G^{kl} + D_\eta G^{kl} = D_y H^{kl} + D_\eta H^{kl}.$$

So by first applying the devices of (7.38) $s - s_l$ times to the l-th term of (7.36), we see that any $D^{s+s_k-1} u^k$ is a sum of terms of the form

$$(7.39) \qquad \int_{E^+} g(x-\xi, y, \eta)\, \phi(\xi,\eta)\, d\xi d\eta \text{ or } \int_{E^+} D_y{}^\nu \omega(x-\xi, y, \eta)\psi(\xi,\eta)\, d\xi d\eta,$$
$$\phi \in C^\mu,\ \psi \in C^{\nu+1+\mu},\ g = D^{s_k+s_l-1} G^{kl},$$

and ω is given by (7.30) and $\phi = 0$ outside G_α.

First, we note that any second derivative of a term of the second type in (7.39) is of the form

$$\int_{E^+} D^2 D_y{}^\nu \omega\,(x-\xi, y, \eta)\,[\psi(\xi,\eta) - \sum_{w=0}^{\nu+1} \nabla^w \psi(x,0)\cdot(\Xi - X)^w / w!]\, d\xi d\eta$$
$$+ \sum_{r+h \leqq \nu+1} [D_y{}^h \psi_{,\beta_1 \cdots \beta_r}(x,0)$$
$$\times \int_{E^+} \xi^{\beta_1} \cdots \xi^{\beta_r} \eta^h D^2 D_y{}^\nu \omega(-\xi, y, \eta)\, d\xi d\eta]/(r+h)!$$

and all the last terms vanish on account of (7.30). The first term in (7.39)

can be written in the form

$$\int_{B_a} \gamma(x-\xi, y-\eta)\phi^*(\xi,\eta)d\xi d\eta - \int_{E^-} \gamma(x-\xi, y-\eta)\phi^*(\xi,\eta)d\xi d\eta$$
$$+ \int_{E^+} h(x-\xi, y, \eta)\phi(\xi,\eta)\,d\xi d\eta = I_1 + I_2 + I_3 \tag{7.40}$$

where $\phi^*(\xi,\eta) = \phi(\xi,\eta)$ if $\eta \geqq 0$ and $\phi^*(\xi,\eta) = \phi(\xi,-\eta)$ if $\eta \leqq 0$ and γ and h are corresponding derivatives of G^{kl} and H^{kl}, respectively. The first term I_1 in (7.40) is of class $C^{1+\mu}$ on $\bar{B}_a$ with desired Hölder bounds on its first derivatives by Lemma 3.7. Next, any second derivative of I_2 in (7.40) involving at least one differentiation with respect to an x^α can be written in the form

$$\int_{E^-} DD_x\gamma(x-\xi, y-\eta)\,[\phi^*(\xi,\eta) - \phi^*(x,y)]\,d\xi d\eta$$

so that

$$|\,DD_xI_2\,| \leqq Z_1(\mu, \nu, N, s_j, m_2, M)\cdot h_\mu(\phi)\cdot y^{\mu-1}. \tag{7.41}$$

In like manner, we see that

$$|\,DD_xI_3\,| \leqq Z_2(\mu, \nu, N, s_j, m_2, M)\cdot h_\mu(\phi)\cdot y^{\mu-1}. \tag{7.42}$$

Thus we see that any derivative $D_xD^{s+s_k-1}u^k \in C^\mu$ for $y \geqq 0$ with

$$h_\mu(D_xD^{s+s_k-1}u^k) \leqq Z_3(\mu, \nu, N, s_j, m_2, M)\sup_j h_\mu(\nabla^{s-s_j}f_j). \tag{7.43}$$

But u is a solution of (5.11) for $y > 0$. The ellipticity of (5.11) allows us to differentiate the j-th equation of (5.11) $s-s_j$ times with respect to y and then solve for $D_y^{s+s_k}u^k$ in terms of the other derivatives so that such derivatives also $\in C^\mu$ and bounds like those in (7.43) hold for those derivatives also.

Now, if each f_j can also be extended to be analytic in x for (x,y) in G_{Rh} with $f_j \in C^{s-s_j+\mu}$ in $\bar{G}_{Rh}$, $0 < R \leqq a$, $0 < h \leqq h_1$, then the u^k can also be extended to be analytic in x for (x,y) on G_{Rh} by merely replacing the part of the range of integration on (ξ,η) for which $(\xi,\eta) \in G_R$ by the manifold $S_{Rh}(x,y)$: η real with $0 < \eta < R$ and ξ on the cone with vertex at x and outer boundary $\partial B_{R'}$ with $R' = (R^2-\eta^2)^{\frac{1}{2}}$ as described in Section 4. Then, for $(x,y) \in G_{hR}$, the analysis of the last two paragraphs can be repeated with the range of integration changed as indicated. Clearly the functions ϕ and ψ of (7.39) are analytic in x for (x,y) in G_{hR} and belong to their respective classes on $\bar{G}_{hR}$ and the function ϕ^* of (7.40) has the same property on the union of G_{hR}, its reflection in $y=0$, and the set $y=0$, $x \in B_{hR}$. Thus the second part of Theorem 7.3 follows.

We can now prove Theorem A′ as stated in Section 5. We construct $P_{R0}(f)$ as follows: Letting (r,σ) be polar coordinates, $0 \leqq r \leqq R$ and σ on the upper unit hemisphere, we define $f^*_j(r,\sigma) = f_j(r,\sigma)$ on the range above, $f^*_j(r,\sigma) = 0$ for $r \leqq 0$, and

$$f^*_j(r,\sigma) = \sum_{q=1}^{s-s_j+1} C_{s-s_j,q} f^*[R - q(r-R)], \qquad r \geqq R,$$

where the C's are determined by the conditions that

$$\sum_{q=1}^{h+1} C_{hq} \cdot (-q)^p = 1, \qquad 0 \leqq p \leqq h.$$

Then $f^*_j \in C^{s-s_j+\mu}$ on E^+ and vanishes outside G_{2R} with $\| f^* \|_{2R} \leqq Z_1(\mu, s_j) \cdot \| f \|$. We let U_R be defined by (7.36) with f_l replaced by f^*_l, let $P_R{}^k$ be the polynomial of degree $s + s_k$ such that $\nabla^p P_R{}^k(0,0) = \nabla^p U_R{}^k(0,0)$ for $0 \leqq p \leqq s + s_k$, and let

$$P_{R0}(f) = u_R \text{ where } u_R{}^k = U_R{}^k - P_R{}^k.$$

Then, clearly $u \in E_{R0}$, $\| u \| \leqq K_q(\mu) \cdot \| f \|$, and it is seen as in Part I that u is a solution of (5.11). Theorem B′ follows from the second part of Theorem 7.3.

To prove Theorem C′, let $H_R \in E_{R0}$ on $\tilde{G}_R$ and be a solution of the homogeneous equations (5.11) on G_R. Let H^*_R be its extension as above, let

$$f^*_{jR} = L_{jk} H^*_R, \qquad U^*_R = P_{R0}(f^*_R), \qquad \mathfrak{H}_R = H^*_R - U^*_R.$$

Then U^*_R, H^*_R, and $\mathfrak{H}_R \in E_{2R0}$ with

$$\| U^*_R \|_{2R} \leqq Z_2 \| H_R \|_R, \qquad \| \mathfrak{H}_R \|_{2R} \leqq Z_3 \| H_R \|_R, \qquad \| H^*_R \|_{2R} \leqq Z_4 \| H_R \|_R$$

and $\mathfrak{H}_R$ is a solution of the homogeneous equations (5.11) on G_{2R}. By applying (7.31) with $u = \mathfrak{H}_R$, $a = 2R$, we obtain

$$\mathfrak{H}_R{}^l(X) = -\int_{\Sigma_{2R}} B^\alpha[G^l(X,\Xi), \mathfrak{H}_R(\Xi)] n_\alpha \, dS_\Xi, \qquad X \in G_{2R} \tag{7.44}$$

since $\mathfrak{H}_R{}^k(\Xi)$ and $G^{lj}(X,\Xi)$ both have 0 Dirichlet data on $\eta = 0$ (see (7.29)). Clearly $\mathfrak{H}_R{}^l(X)$ can be extended analytically in x (X in fact) to any $G_{h,2R} \supset G_{hR}$ with $0 < h \leqq h_1$. Now any derivative $D^{s+s_l+1} \mathfrak{H}_R{}^l(X)$ for $X \in G_{hR}$, $0 < h \leqq h_1$, is seen to be a linear combination of terms of the form

$$I = \int_{\Sigma_{2R}} D_X{}^{s+s_l+1} D_\Xi{}^p G^{lj}(X,\Xi) D_\Xi{}^q \mathfrak{H}_R{}^k(\Xi) n_\alpha \, dS_\Xi, \tag{7.45}$$

$$p + q = s_j + s_k - 1.$$

The term (7.45) may be written in the form

$$I = \int_{\Sigma_{2R}} G(X,\Xi)\,[\phi(\Xi) - \sum_{w=0}^{h} \nabla^w \lambda(X_1)\cdot(\Xi - X_1)^w/w!]\,n_\alpha\,dS_\Xi$$

$$+ \sum_{w=0}^{h} [\nabla^w \phi(X_1) \int_{\Sigma} G(X,\Xi)\,(\Xi - X_1)^w n_\alpha\,dS_\Xi]/w!,$$

$$(h = s - s_j + 1 + p),\ X = X_1 + iX_2 \in G_{hR},$$

where it is clear what G and ϕ stand for. Now, by using only the facts (derivable from homogeneity, etc. and $\mathfrak{H}_R \in E_{2R,0}$) that

$$|\,G(X,\Xi)\,| \leqq Z_1 \cdot \|\,\Xi - X\,\|^{s_j - s - \nu - 2},\quad R \leqq |\,\Xi - X_1\,| \leqq \|\,\Xi - X\,\|;$$

$$|\,\nabla^w \phi(X_1)\,| \leqq \|\,\mathfrak{H}_R\,\|_{2R} \cdot R^{s - s_j + p + 1 - w + \mu},$$

we find that

$$|\,\nabla^{s+s_l+1}\mathfrak{H}_R{}^l(X)\,| \leqq Z_2(h,\mu,\nu,N,s_j,m_2,M)\cdot \|\,H_R\,\|_R \cdot R^{\mu-1},\qquad X \in G_{hR}$$

from which Theorem C′ follows easily since $\|\,H^*{}_R\,\|^*{}_R$ is also $\leqq Z_3 \cdot \|\,H_R\,\|_R$ by Theorem 7.3 since $f^* = 0$ on G_R.

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REFERENCES.

[1] A. Friedman, "Classes of solutions of linear non-hyperbolic differential equations," to appear in *Journal of Mathematics and Mechanics.*

[2] P. Lax and A. N. Milgram, "Parabolic equations," *Annals of Mathematics Studies,* no. 33, Princeton University Press (1954), pp. 167-190.

[3] S. Mandelbrojt, *Séries de Fourier et classes quasi-analytiques de fonctions,* Collection Borel, Paris, 1935.

[4] C. B. Morrey, Jr., "On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations, I," *American Journal of Mathematics,* vol. 80 (1958), pp. 198-218.

[5] L. Nirenberg, "Remarks on strongly elliptic partial differential equations," *Communications on Pure and Applied Mathematics,* vol. 8 (1955), pp. 648-674.

REMARKS ON MY PAPER "THE JACOBIAN VARIETY OF AN ALGEBRAIC CURVE."*

By Wei-Liang Chow.

We should like to make two remarks on this paper, which was published in vol. 76 (1954), pp. 453-76, of this Journal.

(1) In section 5, at the bottom of p. 469, we have stated that if $\mathfrak{P}$ is a divisor class of degree $n > 2g$, rational over an extension K of k, and if $(x^1), \cdots, (x^n)$ are the n points contained in a generic positive divisor $\mathfrak{p}$ of $\mathfrak{P}$ over K, then all the n points $(x^1), \cdots, (x^n)$ are generic points of $\mathfrak{C}$ over K, distinct from each other, and any $n-g$ of these n points are independent with respect to each other over K and determine the remaining g points uniquely. On the basis of this statement it can be easily deduced that there is an automorphism of the field $K((x^1), \cdots, (x^n))$ over the field K which carries any set of $n-g$ of the n points $(x^1), \cdots, (x^n)$ into any other such set; furthermore, it is clear that such an automorphism over K is also an automorphism over $K((p))$, so that the Galois group of $K((x^1)), \cdots, (x^n))$ over $K((p))$ is $(n-g)$-fold transitive. Professor J. Igusa has kindly pointed out to us that it is by no means obvious that any $n-g$ points among the n points $(x^1), \cdots, (x^n)$ are independent with respect to each other over K, and that a formal proof of this statement is desirable, although it is sufficient for the purpose we had there to prove this for the special case where $\mathfrak{p}$ is a generic positive divisor of degree n over k in $\mathfrak{C}$, in which case the proof follows easily from the fact that the Galois group of $k((x^1), \cdots, (x^n))$ over $k((p))$ is the entire symmetric group of permutations of the n points $(x^1), \cdots, (x^n)$. We shall give here a simple proof of our statement as follows.

For any point (η) in $\mathfrak{C}$, we denote by $\mathfrak{C}^n(\eta)$ the subset in $\mathfrak{C}^n$ consisting of all points which represent positive divisors of degree n in $\mathfrak{C}$ having the point (η) as a component. It is clear that $\mathfrak{C}^n(\eta)$ is a subvariety of dimension $n-1$ in $\mathfrak{C}^n$, defined over $k((\eta))$; in fact, if (q) is the point in $\mathfrak{C}^n$ which represents the divisor consisting of the point (η) and $n-1$ independent generic points of $\mathfrak{C}$ over $k((\eta))$, then $\mathfrak{C}^n(\eta)$ is the locus of (q)

* Received December 1, 1957.

over $k((\eta))$. Furthermore, it is easily seen that if (x) is a generic point of $\mathfrak{C}$ over k, then $\mathfrak{C}^n(\eta)$ is the unique specialization of $\mathfrak{C}^n(x)$ over the specialization $(x) \to (\eta)$ over k. In order to prove our statement, it is sufficient to show that if the $n-g$ points $(x^1), \cdots, (x^{n-g})$ in $\mathfrak{p}$ are independent generic points of $\mathfrak{C}$ over K, then the same is true for the $n-g$ points $(x^1), \cdots, (x^{n-g-1}), (x^s)$, for any choice of the point (x^s) among the remaining g points in $\mathfrak{p}$ (always under the assumption $n > 2g$). To show this, we shall assume that it is possible to choose the point (x^s) so that it is algebraic over $K((x^1), \cdots, (x^{n-g-1}))$, and we shall show that this leads to a contradiction. Since $(x^1), \cdots, (x^{n-g-1})$ are independent generic points of $\mathfrak{C}$ over K, the set of all positive divisor in the residual class of $\mathfrak{P}$ with respect to the divisor $\sum_{i=1}^{n-g-1} (x^i)$ is a complete linear system of dimension 1, rational over $K((x^1), \cdots, (x^{n-g-1}))$, and hence is represented in $\mathfrak{C}^{g+1}$ by a subvariety $\mathfrak{E}$ of dimension 1, defined over $K((x^1), \cdots, (x^{n-g-1}))$. The assumption that (x^s) is algebraic over $K((x^1), \cdots, (x^{n-g-1}))$ implies that (x^s) is a fixed point of this complete linear system, so that we have the relation $\mathfrak{E} \subset \mathfrak{C}^{g+1}(x^s)$. Consider now the simultaneous specialization $(x^i) \to (x^{n-i+1})$, for $i = 1, \cdots, n-g-1$, over K, and let $\mathfrak{E}'$ and (ξ) be specializations of the positive 1-cycle $\mathfrak{E}$ and the point (x^s) over this specialization over K; then every component $\mathfrak{E}_0'$ in $\mathfrak{E}'$ is a subvariety of dimension 1 and we have the relation $\mathfrak{E}_0' \subset \mathfrak{C}^{g+1}(\xi)$. If (q) is any point in $\mathfrak{E}_0'$ and $\mathfrak{q}$ is the positive divisor in $\mathfrak{C}$ represented by (q), then the positive divisor $\mathfrak{q} + \sum_{i=1}^{n-g-1} (x^{n-i+1})$ is a specialization over K of a positive divisor in $\mathfrak{P}$ and hence is contained in $\mathfrak{P}$; it follows that every such positive divisor $\mathfrak{q}$ is an element in the residual class of $\mathfrak{P}$ with respect to the divisor $\sum_{i=1}^{n-g-1} (x^{n-i+1})$. Now, the set of all positive divisors in this last mentioned residual class is the complete linear system determined by the positive divisor $\sum_{i=1}^{g+1} (x^i)$ and since $n-g \geqq g+1$, the points $(x^1), \cdots, (x^{g+1})$ are independent generic points of $\mathfrak{C}$ over K, and hence the divisor $\sum_{i=1}^{g+1} (x^i)$ is a generic positive divisor of degree $g+1$ over K in $\mathfrak{C}$. This implies in particular that this complete linear system has the dimension 1 and hence is represented in $\mathfrak{C}^{g+1}$ by a subvariety $\mathfrak{E}''$ of dimension 1; since $\mathfrak{E}_0' \subset \mathfrak{E}''$ and both varieties have the same dimension, we conclude that $\mathfrak{E}_0' = \mathfrak{E}''$ and hence $\mathfrak{E}'' \subset \mathfrak{C}^{g+1}(\xi)$. This means that the complete linear system determined by the generic positive divisor $\sum_{i=1}^{g+1} (x^i)$ of degree $g+1$ over K in $\mathfrak{C}$ has a fixed point, which is evidently impossible.

(2) In the last paragraph of our paper, on p. 475, we have given a proof of the statement that if the curve $\mathfrak{C}$ contains a rational divisor $\mathfrak{o}$ over k of degree 1, then every rationally defined divisor class over any extension K of k is also rational over K. Dr. P. Roquette has kindly called our attention to the fact that the proof given there holds only for the case where $\mathfrak{o}$ is a positive divisor, or rather a prime divisor, so that there is in this sense a gap in our proof. However, this gap can be easily remedied as follows. In fact, it is sufficient to prove the following statement: If $\mathfrak{C}$ contains a rational positive divisor $\mathfrak{o}$ over k of degree s and if $\mathfrak{P}$ is a rationally defined divisor class over K, then the divisor class $s\mathfrak{P}$ is rational over K. Now, if $(x^1), \cdots, (x^s)$ are the s points (which need not be distinct) contained in $\mathfrak{o}$, then, by what we have proved there, there exists a rational divisor $\mathfrak{p}_i$ over $K((x^i))$ in $\mathfrak{P}$ for each $i = 1, \cdots, s$; the divisor $\sum_{i=1}^{s} \mathfrak{p}_i$ is evidently rational over K and is contained in sP.

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[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Societé Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

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